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On Mathematical Aspects of Dual Variables in Continuum Mechanics. Part 1: Mathematical Principles

Dedicated to Professor Dr.-Ing. Dr.-Ing. E. h. Dr. h.c. mult. E. STEIN on the occasion of his 65th birthday

In diesem, aus zwei Teilen bestehendem Aufsatz werden mathematische Gesichtspunkte von dualen Variablen behandelt, die in der Kontinuumsmechanik auftreten. Dabei wird der Tensorkalkül auf Mannigfaltigkeiten verwendet, wie er von MARSDEN *und* HUGHES [1] *in die Kontinuumsmechanik eingeführt wurde. Dieser mathematische Formalismus führt zu zusätzlicher Struktur in kontinuumsmechanischen Theorien. Insbesondere ergibt die Invarianz bestimmter Bilinearformen eindeutige Transformationsregeln für Tensoren zwischen der Referenz- und der Momentankonfiguration. Diese Transformationsregeln werden durch die push-forward- bzw. pull-back-Operatoren festgelegt. – In Teil 1 stellen wir die mathematischen Grundlagen unseres Vorgehens vor. Ein wesentlicher Aspekt besteht darin, sorgfältig zwischen inneren und skalaren Produkten zu unterscheiden. Diese Unterscheidung wird physikalisch motiviert und mathematisch formuliert. Innere Produkte können nur für solche Objekte gebildet werden, die in ein und demselben Vektorraum leben. Dagegen werden Skalarprodukte aus Objekten gebildet, die in verschiedenen Vektorräumen leben. Die Unterscheidung von inneren und skalaren Produkten führt zu einer Unterscheidung zwischen transponierten und dualen Tensoren. Entsprechend wird zwischen Symmetrie und Selbstdualität unterschieden. Ein wichtiges Ergebnis der Untersuchungen sind neue Beziehungen für die Berechnung des push-forwards bzw. pull-backs von Tensoren zweiter Stufe. Sie werden aus Invarianzforderungen für bestimmte innere bzw. skalare Produkte hergeleitet. Im Gegensatz zu den aus der Literatur bekannten Beziehungen bleibt bei diesen Formeln die Symmetrie gemischter Tensoren erhalten.*

In this paper consisting of two parts we consider mathematical aspects of dual variables appearing in continuum mechanics. Tensor calculus on manifolds as introduced into continuum mechanics by MARSDEN *and* HUGHES [1] *is used as a point of departure. This mathematical formalism leads to additional structure of continuum mechanical theories. Specifically invariance of certain bilinear forms renders unambiguous transformation rules for tensors between the reference and the current configuration. These transformation rules are determined by push-forwards and pull-backs, respectively. – In Part 1 we consider the basic mathematical features of our theory. The key aspect of our approach is that, contrary to the usual considerations in this field, we distinguish carefully between inner products and scalar products. This discrimination is motivated by physical considerations and is subsequently given a firm mathematical basis. Inner products can only be formed with objects living in one and the same vector space. Scalar products, on the other hand, are formed between objects living in different spaces. The distinction between inner and scalar products leads to a distinction between transposes and duals of tensors. Therefore, we distinguish between symmetry and self-duality. An important result of this approach are new formulae for the computation of push-forwards and pull-backs, respectively, of second-order tensors, which are derived from invariance requirements of inner and scalar products, respectively. In contrast to prior approaches these new formulae preserve symmetry of symmetric mixed tensors.*

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1. Introduction

Tensor calculus on manifolds is a powerful tool for the mathematical description of continuum mechanics. Prior mathematical formulations do not carry through the consequences of physical duality. The objective of this paper is to formulate a description in which duality is attributed a key mathematical role.

In Part 1 of our paper we present the mathematical fundamentals of tensor calculus on manifolds with respect to applications in continuum mechanics. In a following Part 2 we give applications of our formalism on nonlinear solid mechanics. The references for both parts of this paper will be given at the end of Part 1.

The issue of dual variables with respect to work or power is of fundamental importance in all fields of physics. In continuum mechanics, conjugate or dual variables are quantities that leave the stress power per unit volume invariant under transformations between different configurations of a material body. Since HILL's key paper [2], this issue has been addressed repeatedly by many authors up until today. Among the more recent studies, we mention the work by HAUPT and TSAKMAKIS [3] who stipulated that dual variables and their suitably defined time derivatives leave invariant the following four *inner products* on different configurations: Inner product of dual stress and strain tensors, stress power, complementary stress power and incremental stress power. Following a different line, SANSOUR [4] states that the derivative of the free energy function with respect to a strain tensor delivers its dual stress tensor. This is similar to a point of view raised many years before by BESSELING [5]. This definition implies that the material response of the body considered is, at least partially, hyperelastic. However, we feel that the concept of dual variables shall be addressed without recurrence to a specific material behavior.

The literature cited so far has been formulated within the framework of classical tensor calculus in a three-dimensional Euclidean space. The concept of tensor calculus on manifolds as introduced into continuum mechanics by MARSDEN and HUGHES [1] has not been considered. It is well known that despite its rather abstract level this mathematical formalism leads to substantial additional structure of continuum mechanical theories. One essential task of continuum mechanical theories is the transformation of objects (as e.g. vectors and tensors of arbitrary order) between the reference and the current configuration, respectively. In classical tensor calculus the transformation rules for e.g. second order tensors with covariant, mixed or contravariant component representations are different and not intelligible directly. In tensor calculus on manifolds the notions of push-forwards and pull-backs lead automatically to the correct transformation rules. Also the concept of Lie derivatives renders time derivatives which are objective. These examples demonstrate the power of tensor calculus on manifolds for continuum mechanics. Therefore, in this paper we will combine tensor calculus on manifolds systematically with the concept of duality. This approach leads to a theory with additional structure thus clarifying basic concepts.

In an abstract setting, all physical quantities can be considered as elements of suitably defined vector spaces. Variables that, for physical reasons, identify different physical quantities should be placed, mathematically, in different spaces. Thus, physically dual variables live in different vector spaces. This becomes simply evident in the mechanics of mass points. Consider a force $\vec{\mathbf{f}}$ acting on an unconstrained mass point. Under the action of this force the mass point is accelerated and sees a velocity $\vec{\mathbf{v}}$. Therefore, the force extends power P to the mass point, where in conventional mechanics this power is defined as the inner product of the vectors $\vec{\mathbf{f}}$ and $\vec{\mathbf{v}}$, i.e. $P = \vec{\mathbf{f}} \cdot \vec{\mathbf{v}}$, where the dot $(\cdot)$ indicates the inner product of classical vector calculus. Mathematically, this formula implies that the force $\vec{\mathbf{f}}$ and the velocity $\vec{\mathbf{v}}$ live in one and the same vector space. However, physically, force and velocity have different dimensions. Since in any vector space the usual vector operations, like addition, must hold for all its elements, it is evident that a clear distinction between forces and velocities has to be made and that these quantities cannot live in one and the same vector space.

Of course, one can reduce all physical quantities to nondimensional numbers by normalizing them. Such nondimensional quantities then formally live in the same vector spaces (for instance normalized vectors in $\mathbb{R}^3$). However, normalized vectors representing different physical quantities (as e.g. force and velocity) must not be added. Therefore, they cannot be elements of one and the same vector space. They rather must be assigned to different vector spaces. The far reaching duality between kinematic and dynamic variables (compare Sections 7 and 8) becomes most noticeable mathematically in our approach.

The mathematical tool to express such a physically based distinction is the concept of *dual spaces* as introduced in the literature (e.g. [6]). In such a setting, we can consider velocities to be elements of a primal vector space $\mathscr{V}$ and forces as elements of a dual vector space $\mathscr{V}^*$, where duality of vector spaces has still to be defined. Further, if we want to measure the magnitude of elements of these vector spaces we have to stipulate that the spaces $\mathscr{V}$ and $\mathscr{V}^*$ are *inner product spaces.* With $\vec{\mathbf{u}} \in \mathscr{V}$, the magnitude $|\vec{\mathbf{u}}|$ of $\vec{\mathbf{u}}$ can be defined as the square root of its inner product, i.e. $|\vec{\mathbf{u}}| = (\vec{\mathbf{u}} \cdot \vec{\mathbf{u}})^{1/2}$. We emphasize the fact that the inner product is a map of the Cartesian product $\mathscr{V} \times \mathscr{V}$ on the space $\mathbb{R}$ of real numbers. Correspondingly, the magnitude of the force $\vec{\mathbf{f}} \in \mathscr{V}^*$ can be measured as square root of the inner product $|\vec{\mathbf{f}}| = (\vec{\mathbf{f}} \cdot \vec{\mathbf{f}})^{1/2}$, where now the inner product is a map of the Cartesian product $\mathscr{V}^* \times \mathscr{V}^*$ on $\mathbb{R}$.

If we now consider the power exerted by the force $\vec{\mathbf{f}}$ it becomes obvious that we have to map the Cartesian product of the velocity $\vec{\mathbf{v}} \in \mathscr{V}$ and its dual, i.e. the force $\vec{\mathbf{f}} \in \mathscr{V}^*$ on the set of real numbers. Since $\vec{\mathbf{v}}$ and $\vec{\mathbf{f}}$ live in different vector spaces no inner product can be defined. This conflict can be resolved by the introduction of a *scalar product.* A scalar product can be defined as a bilinear map of the Cartesian product $\mathscr{V} \times \mathscr{V}^*$ on the set $\mathbb{R}$ of real numbers. VAN DER GIESSEN [7] has given a preliminary formulation of a continuum mechanics based on such considerations. In the present paper, these ideas are developed further within the framework of tensor calculus on manifolds [1].

The layout of the whole paper is as follows. In Part 1 we summarize in Section 2 the development of tensor algebra on a manifold in which inner and scalar products, and spaces and dual spaces are kept separately. For conciseness, the exposition focusses exclusively on the type of tensors that appear typically in continuum mechanics. Section 3 deals with tensor operations that are relevant when considering maps between manifolds. Particular emphasis in this section will be on the development of the concepts of pull-backs and push-forwards within the novel framework adopted here. In Part 2 we first recapitulate in Section 5 the essential results of Part 1. In Sections 6 and 7 we discuss the applications of the present approach to the kinematics and dynamics of continuous media. Section 8 closes with a few illustrative examples from the field of constitutive modelling which demonstrate some of the illuminating features of the formulation.

2 Algebra of tensors on a manifold

2.1 Vectors and one-forms

Throughout this paper the summation convention is used. Following TRUESDELL and NOLL [8] we consider a body B with a configuration $\mathscr{B}$ being an n-dimensional differentiable manifold, where n can take values 1 or 2 or 3. Let $\mathscr{U} \subset \mathscr{B}$ be an open set. Further, let $\{X^A\} : \mathscr{U} \to \mathbb{R}^n$, $A = 1, 2, \ldots, n$, be a coordinate system on $\mathscr{B}$. Then we can introduce covariant basis vectors $\vec{\mathbf{E}}_A$ on $\mathscr{B}$ as

$$\vec{\mathbf{E}}_A := \partial/\partial X^A\,, \qquad A = 1, 2, \ldots, n\,. \tag{1}$$

Since $\mathbb{R}^n$ is a linear vector space, and since the manifold $\mathscr{B}$ is embedded in $\mathbb{R}^n$ ($\mathscr{B} \subset \mathbb{R}^n$) we can attach to each element $X \in \mathscr{B}$ the linear vector space $\mathbb{R}^n$. Let $\vec{\mathbf{V}} \in \mathbb{R}^n$ be a typical vector of this space. Then we can define the *tangent space* at the point $X \in \mathscr{B}$.

Definition 2.1: The *tangent space $T_X\mathscr{B}$ at a point* $X \in \mathscr{B}$ is the linear vector space of all vectors $\vec{\mathbf{V}} \in \mathbb{R}^n$,

$$\vec{\mathbf{V}} = V^A \vec{\mathbf{E}}_A\,, \qquad V^A \in \mathbb{R}\,, \tag{2}$$

emanating at the point $X \in \mathscr{B}$. □

Remark 2.1: We denote all vectors defined on $T_X\mathscr{B}$ by uppercase boldface Roman letters with a superposed arrow, e.g. $\vec{\mathbf{U}}$.

Definition 2.2: A *vector field* $\vec{\mathbf{V}}(X)$ on the manifold $\mathscr{B}$ is a map

$$\vec{\mathbf{V}}(X) : \mathscr{B} \to T_X\mathscr{B} \tag{3}$$

for all $X \in \mathscr{B}$. □

Next, we introduce the notion of *cotangent space.* For this purpose we consider the space $\mathscr{L}(T_X\mathscr{B};\, \mathbb{R})$ of all linear functionals $f : T_X\mathscr{B} \to \mathbb{R}$. Since $\mathbb{R}$ is a linear vector space itself, we can conclude from theorem 16.1 in BOWEN and WANG [6] that

$$\dim \mathscr{L}(T_X\mathscr{B};\, \mathbb{R}) = \dim T_X\mathscr{B} \dim \mathbb{R} = \dim T_X\mathscr{B} = n\,.$$

Therefore, the spaces $T_X\mathscr{B}$ and $\mathscr{L}(T_X\mathscr{B};\, \mathbb{R})$ are isomorphic and $\mathscr{L}(T_X\mathscr{B};\, \mathbb{R})$ is itself a vector space called the cotangent space of the manifold $\mathscr{B}$ at the point $X \in \mathscr{B}$.

Definition 2.3: The *cotangent space* $T^*_X\mathscr{B}$ is the linear vector space $\mathscr{L}(T_X\mathscr{B};\, \mathbb{R})$ of all linear maps $f : T_X\mathscr{B} \to \mathbb{R}$ emanating at $X \in \mathscr{B}$, i.e.

$$T^*_X\mathscr{B} = \mathscr{L}(T_X\mathscr{B};\, \mathbb{R}) \quad \square\,. \tag{4}$$

The tangent space $T_X\mathscr{B}$ and the cotangent space $T^*_X\mathscr{B}$ are *dual spaces.* The elements of the cotangent space are called *one-forms.* (Sometimes one-forms are also called *covectors.*)

Remark 2.2: We denote one-forms living in the dual space $T^*_X\mathscr{B}$ by boldface Greek letters with a superposed arrow, e.g. $\vec{\boldsymbol{\alpha}} \in T^*_X\mathscr{B}$. □

Similar to Definition 2.2, we can define one-form fields on $\mathscr{B}$. Remember that a one-form is a linear function from $T_X\mathscr{B}$ into $\mathbb{R}$, i.e.

$$\vec{\boldsymbol{\alpha}} : T_X\mathscr{B} \to \mathbb{R} : \vec{\mathbf{V}} \mapsto \vec{\boldsymbol{\alpha}}(\vec{\mathbf{V}}) = k\,, \tag{5}$$

where $\vec{\mathbf{V}} \in T_X\mathscr{B}$ and $k \in \mathbb{R}$. Let $m,\, n \in \mathbb{R}$ and $\vec{\mathbf{U}},\, \vec{\mathbf{V}} \in T_X\mathscr{B}$ and $\vec{\boldsymbol{\alpha}} \in T^*_X\mathscr{B}$. Then we have

$$\vec{\boldsymbol{\alpha}}(m\vec{\mathbf{U}} + n\vec{\mathbf{V}}) = m\vec{\boldsymbol{\alpha}}(\vec{\mathbf{U}}) + n\vec{\boldsymbol{\alpha}}(\vec{\mathbf{V}})\,, \tag{6}$$

since $\vec{\boldsymbol{\alpha}}$ is a linear function from $T_X\mathscr{B}$ into $\mathbb{R}$. Correspondingly, let $\vec{\boldsymbol{\alpha}},\, \vec{\boldsymbol{\beta}} \in T^*_X\mathscr{B}$ and $\vec{\mathbf{U}} \in T_X\mathscr{B}$. Then from linearity of the maps $\vec{\boldsymbol{\alpha}} : T_X\mathscr{B} \to \mathbb{R}$ and $\vec{\boldsymbol{\beta}} : T_X\mathscr{B} \to \mathbb{R}$ follows

$$(m\vec{\boldsymbol{\alpha}} + n\vec{\boldsymbol{\beta}})\, \vec{\mathbf{U}} = m\vec{\boldsymbol{\alpha}}(\vec{\mathbf{U}}) + n\vec{\boldsymbol{\beta}}(\vec{\mathbf{U}})\,. \tag{7}$$

Therefore, the action of a one-form on a vector is a bilinear form. We denote this bilinear form by $\langle .\,,\, . \rangle_X$ and call it a *scalar product* [6].

Definition 2.4: A *scalar product*

$$\langle \vec{\boldsymbol{\alpha}},\, \vec{\mathbf{V}} \rangle_X : T^*_X\mathscr{B} \times T_X\mathscr{B} \to \mathbb{R} \tag{8}$$

is a bilinear mapping of the Cartesian product $T^*_X\mathscr{B} \times T_X\mathscr{B}$ on the set of real numbers $\mathbb{R}$, such that

$$\langle \vec{\boldsymbol{\alpha}},\, \vec{\mathbf{V}} \rangle_X := \vec{\boldsymbol{\alpha}}(\vec{\mathbf{V}})\,. \quad \square \tag{9}$$

Remark 2.3: We denote scalar products between objects living in the spaces $T_X\mathscr{B}$ and $T^*_X\mathscr{B}$, respectively, by $\langle .\,,\, . \rangle_X$, where the index X indicates that the dual spaces are attached to the manifold $\mathscr{B}$ at $X \in \mathscr{B}$. □

Remark 2.4: For all intents and purposes, the dual of the cotangent space $T^*_X\mathscr{B}$ is identical to the tangent space $T_X\mathscr{B}$, i.e. $T_X\mathscr{B} = \mathscr{L}(T^*_X\mathscr{B};\, \mathbb{R})$. Hence, a vector $\vec{\mathbf{V}}_X\mathscr{B}$ can be considered as a linear function

$$\vec{\mathbf{V}} : T^*_X\mathscr{B} \to \mathbb{R} : \vec{\boldsymbol{\alpha}} \mapsto V(\vec{\boldsymbol{\alpha}})\,, \tag{10}$$

while linearity implies that $\vec{\mathbf{V}}(\vec{\boldsymbol{\alpha}}) = \vec{\boldsymbol{\alpha}}(\vec{\mathbf{V}})$. Because of the mutual duality between $T_X\mathscr{B}$ and $T^*_X\mathscr{B}$, the definition of the scalar product in (8) can be extended to handle bilinear maps from $T_X\mathscr{B} \times T^*_X\mathscr{B}$ to $\mathbb{R}$,

$$\langle \vec{\mathbf{V}},\, \vec{\boldsymbol{\alpha}} \rangle_X : T_X\mathscr{B} \times T^*_X\mathscr{B} \to \mathbb{R}\,, \tag{11}$$

so that the scalar product acts as a *symmetric* bilinear form, $\langle \vec{\mathbf{V}},\, \vec{\boldsymbol{\alpha}} \rangle_X = \langle \vec{\boldsymbol{\alpha}},\, \vec{\mathbf{V}} \rangle_X$.

Thus, the order of the spaces in the Cartesian product on which the scalar product is formed is of no importance, but it is crucial for our considerations that scalar products map Cartesian products of dual spaces on the set of real numbers. □

Using the notion of scalar product we can introduce a basis on the cotangent space $T^*_X\mathscr{B}$. This basis is called the *dual basis* and we denote it as $\vec{\boldsymbol{\varepsilon}}^A$, $A = 1, 2 \ldots, n$. The dual basis is given by

$$\langle \vec{\boldsymbol{\varepsilon}}^A, \vec{\mathbf{E}}_B \rangle_X = \delta^A_B , \tag{12}$$

where δ^A_B is the Kronecker symbol. It can be shown [1] that for a coordinate system $\{X^A\}$ the dual basis is given by

$$\vec{\boldsymbol{\varepsilon}}^A = \mathrm{d}X^A . \tag{13}$$

The component representation of a one-form is given by

$$\vec{\boldsymbol{\alpha}} = \alpha_A \vec{\boldsymbol{\varepsilon}}^A , \qquad \alpha_A \in \mathbb{R} . \tag{14}$$

From (2), (14), and (12), it follows that

$$\langle \vec{\boldsymbol{\alpha}}, \vec{\mathbf{V}} \rangle_X = \langle \alpha_A \vec{\boldsymbol{\varepsilon}}^A, V^B \vec{\mathbf{E}}_B \rangle_X = \alpha_A V^A . \tag{15}$$

Therefore, the scalar product is an example of a product leading to a simple contraction of components.

Remark 2.5: We emphasize that, contrary to classical tensor calculus, the dual basis does not consist of basis "vectors" but rather of basis one-forms. □

Remark 2.6: In conformance with Remark 2.2 we reserve the kernel letter $\vec{\boldsymbol{\varepsilon}}$ for the dual basis on the manifold $\mathscr{B}$. □

2.2 Tensors on manifolds

We can generalize the concepts of vectors and one-forms, respectively, to tensors. For this purpose, we need the following results from combinatorics. Consider two arbitrary elements A and B.

Definition 2.5: A (p, q)-*class* is the set consisting of all permutations of p elements A and q elements B. The number of permutations within a (p, q)-class is equal to $(p+q)!$. □

However, by simple inspection it is observed that within a (p, q)-class there exist $p!q!$ permutations that are indistinguishable because they are simple permutations of the elements A and B, i.e., elements A are exchanged with elements A and elements B are exchanged with elements B, respectively. Now it is possible to sort out from a (p, q)-class the subclass of *distinguishable* permutations of the p elements A and the q elements B. Such distinguishable permutations are generated from an arbitrary representative of a (p, q)-class by exchanging elements A with elements B and vice versa.

Definition 2.6: A (p, q)-*restriction* is the set of all distinguishable permutations of p elements A and q elements B. Let N be the number of all sets in a (p, q)-restriction. Then

$$N = \frac{(p+q)!}{p!q!} \quad \square \tag{16}$$

Now we can define the $\binom{p}{q}$-family of *associate tensors*. For this purpose we identify the element A with the cotangent space $T^*_X\mathscr{B}$ and the element B with the tangent space $T_X\mathscr{B}$. Next, we form Cartesian products consisting of p elements in $T^*_X\mathscr{B}$ and q elements in $T_X\mathscr{B}$. From the class of all such Cartesian products we sort out the subclass of distinguishable Cartesian products.

Definition 2.7: The $\binom{p}{q}$-*restriction* of Cartesian products of p elements in $T^*_X\mathscr{B}$ and q elements in $T_X\mathscr{B}$ is the (p, q)-restriction of all such Cartesian products. □

For the definition of tensors, the notion of *multilinear mappings* is of central importance.

Definition 2.8: The $\binom{p}{q}$-*family of associate tensors* is the set of all multilinear mappings of the $\binom{p}{q}$-restriction of Cartesian products of p elements in $T^*_X\mathscr{B}$ and q elements in $T_X\mathscr{B}$ on the set $\mathbb{R}$ of real numbers.

A *tensor of type* $\binom{p}{q}$ is an individual member of the $\binom{p}{q}$-family of associate tensors.

Definition 2.9: A *tensor of type* $\binom{p}{q}$ is a multilinear mapping of a Cartesian product of p elements in $T^*_X\mathscr{B}$ and q elements in $T_X\mathscr{B}$ on the set $\mathbb{R}$ of real numbers. Therefore, a tensor of type $\binom{p}{q}$ is a representative of the $\binom{p}{q}$-family of associate tensors. Any tensor of type $\binom{p}{q}$ is *contravariant* of rank p and *covariant* of rank q. □

We demonstrate these ideas for the $\binom{2}{2}$-family of associate tensors. From (16) it follows that this family consists of 6 associate tensors, the definitions of which read as follows:

$$\mathbf{T}^{\flat\sharp} : T^*_X\mathscr{B} \times T^*_X\mathscr{B} \times T_X\mathscr{B} \times T_X\mathscr{B} \to \mathbb{R}\,, \tag{17}$$

$$\mathbf{T}^{//} : T^*_X\mathscr{B} \times T_X\mathscr{B} \times T^*_X\mathscr{B} \times T_X\mathscr{B} \to \mathbb{R}\,, \tag{18}$$

$$\mathbf{T}^{/\backslash} : T^*_X\mathscr{B} \times T_X\mathscr{B} \times T_X\mathscr{B} \times T^*_X\mathscr{B} \to \mathbb{R}\,, \tag{19}$$

$$\mathbf{T}^{\backslash\backslash} : T_X\mathscr{B} \times T^*_X\mathscr{B} \times T_X\mathscr{B} \times T^*_X\mathscr{B} \to \mathbb{R}\,, \tag{20}$$

$$\mathbf{T}^{\backslash/} : T_X\mathscr{B} \times T^*_X\mathscr{B} \times T^*_X\mathscr{B} \times T_X\mathscr{B} \to \mathbb{R}\,, \tag{21}$$

$$\mathbf{T}^{\sharp\flat} : T_X\mathscr{B} \times T_X\mathscr{B} \times T^*_X\mathscr{B} \times T^*_X\mathscr{B} \to \mathbb{R}\,. \tag{22}$$

For an explanation of symbols $^\backslash, ^/$, $\sharp$, and $\flat$ see Remark 2.13.

Remark 2.7: We note that contrary to classical tensor calculus, all associate tensors of a $\binom{p}{q}$-family are different tensors since they represent different mappings. □

Remark 2.8: We denote tensors defined on $\mathscr{B}$ by boldface uppercase Roman letters. □

Since it is not possible to consider in the sequel all members of a $\binom{p}{q}$-family of associate tensors, we shall concentrate in an exemplaric manner on the following member of the $\binom{p}{q}$-family, which we denote as the *representative* tensor.

Definition 2.10: The *representative tensor* $\mathbf{T}$ of a $\binom{p}{q}$-family of associate tensors at the point $X \in \mathscr{B}$ is the multilinear mapping

$$\mathbf{T} : (\underbrace{T^*_X\mathscr{B} \times \ldots \times T^*_X\mathscr{B}}_{p \text{ copies}}) \times (\underbrace{T_X\mathscr{B} \times \ldots \times T_X\mathscr{B}}_{q \text{ copies}}) \to \mathbb{R}\,. \tag{23}$$

The *components* of $\mathbf{T}$ are defined as

$$\mathrm{T}^{A_1A_2\ldots A_p}{}_{B_1B_2\ldots B_q} = \mathbf{T}(\vec{\boldsymbol{\varepsilon}}^{A_1}, \vec{\boldsymbol{\varepsilon}}^{A_2}, \ldots, \vec{\boldsymbol{\varepsilon}}^{A_p}, \vec{\mathbf{E}}_{B_1}, \vec{\mathbf{E}}_{B_2}, \ldots, \vec{\mathbf{E}}_{B_q})\,. \quad \square \tag{24}$$

Since the representative tensor $\mathbf{T}$ is a member of the $\binom{p}{q}$-family of associate tensors it is contravariant of rank p and covariant of rank q (compare Definition 2.9). The action of a tensor $\mathbf{T}$ of type $\binom{p}{q}$ on p one-forms and q vectors gives a real number, i.e.

$$\mathbf{T}(\vec{\boldsymbol{\alpha}}^1, \ldots, \vec{\boldsymbol{\alpha}}^p, \vec{\mathbf{V}}_1, \ldots, \vec{\mathbf{V}}_q) = \mathrm{T}^{A_1A_2\ldots A_p}{}_{B_1B_2\ldots B_q}\alpha^1_{A_1}\ldots\alpha^p_{A_p}V_1^{B_1}\ldots V_q^{B_q}\,, \tag{25}$$

where $\vec{\boldsymbol{\alpha}}^i = \alpha^i_A\vec{\boldsymbol{\varepsilon}}^A \in T^*_X\mathscr{B}$ and $\vec{\mathbf{V}}_j = V_j^B\vec{\mathbf{E}}_B \in T_X\mathscr{B}$. Clearly, a vector is a $\binom{1}{0}$-tensor and a one-form a $\binom{0}{1}$-tensor. A scalar can be interpreted as a $\binom{0}{0}$-tensor.

Definition 2.11: The *order of a* $\binom{p}{q}$*-tensor* is defined as

$$O := p + q\,. \quad \square \tag{26}$$

Generalizing Definition 2.2 we can define $\binom{p}{q}$-tensor fields on $\mathscr{B}$, where again this notion is introduced for the representative tensors only.

Definition 2.12: The set of all representative tensors of a $\binom{p}{q}$-family of associate tensors forms the *tensor space* $\mathscr{T}^p_q$ defined by

$$\mathscr{T}^p_q = \mathscr{L}(\underbrace{T^*_X\mathscr{B} \times \ldots \times T^*_X\mathscr{B}}_{p \text{ copies}} \times \underbrace{T_X\mathscr{B} \times \ldots \times T_X\mathscr{B}}_{q \text{ copies}}; \mathbb{R})\,. \quad \square \tag{27}$$

Remark 2.9: In Definition 2.12 we again used the notion of a representative tensor (see Definition 2.10). It should be noted that to each tensor of type $\binom{p}{q}$ belonging to the $\binom{p}{q}$-family of associate tensors, an individual tensor space has to be assigned, as will be demonstrated in Definition 2.15 or Definition 2.16. □

Definition 2.13: A $\binom{p}{q}$-*tensor field on the manifold* $\mathscr{B}$ is a map

$$\mathbf{T}(X) : X \to \mathscr{T}^p_q \tag{28}$$

for all $X \in \mathscr{B}$. □

To emphasize that $\mathscr{T}^p_q$ is defined on the tangent space $T_X\mathscr{B}$ and the cotangent space $T^*_X\mathscr{B}$, respectively, we may adopt the notation $\mathscr{T}^p_q(T_X\mathscr{B}, T^*_X\mathscr{B})$, but we shall refrain from that since there is no possibility for confusion.

Any tensor $\mathbf{T} \in \mathscr{T}^p_q$ can be represented by a basis formed by tensor products of one-forms and vectors.

Definition 2.14: Let $\mathbf{T}_1 \in \mathscr{T}^p_q$ and $\mathbf{T}_2 \in \mathscr{T}^r_s$. Then the *tensor product* is defined by

$$\begin{aligned}&(\mathbf{T}_1 \otimes \mathbf{T}_2)\,(\vec{\boldsymbol{\alpha}}^1, \ldots, \vec{\boldsymbol{\alpha}}^p, \vec{\boldsymbol{\beta}}^1, \ldots, \vec{\boldsymbol{\beta}}^r, \vec{\mathbf{U}}_1, \ldots, \vec{\mathbf{U}}_q, \vec{\mathbf{V}}_1, \ldots, \vec{\mathbf{V}}_s)\\ &= \mathbf{T}_1(\vec{\boldsymbol{\alpha}}^1, \ldots, \vec{\boldsymbol{\alpha}}^p, \vec{\mathbf{U}}_1, \ldots, \vec{\mathbf{U}}_q)\,\mathbf{T}_2(\vec{\boldsymbol{\beta}}^1, \ldots, \vec{\boldsymbol{\beta}}^r, \vec{\mathbf{V}}_1, \ldots, \vec{\mathbf{V}}_s)\,,\end{aligned} \tag{29}$$

where $\vec{\boldsymbol{\alpha}}^i, \vec{\boldsymbol{\beta}}^j \in T^*_X\mathscr{B}$, with $i = 1, 2, \ldots, p$, $j = 1, 2 \ldots, r$, and $\vec{\mathbf{U}}_k, \vec{\mathbf{V}}_l \in T_X\mathscr{B}$ with $k = 1, 2, \ldots, q$, $l = 1, 2, \ldots, s$.

The space of all such tensors $\mathbf{T}_1 \otimes \mathbf{T}_2$ is the tensor space $\mathscr{T}^{p+r}_{q+s} = \mathscr{T}^p_q \otimes \mathscr{T}^r_s$. It follows, for instance, that $\mathscr{T}^1_1 = \mathscr{L}(T^*_X\mathscr{B}, T_X\mathscr{B}; \mathbb{R})$ can also be written as $\mathscr{T}^1_1 = \mathscr{T}^1_0 \otimes \mathscr{T}^0_1 = T_X\mathscr{B} \otimes T^*_X\mathscr{B}$. More general, $\mathscr{T}^p_q$ itself can be viewed as the tensor product of $T_X\mathscr{B}$ and $T^*_X\mathscr{B}$ [see (27)], i.e.

$$\mathscr{T}^p_q = \underbrace{T_X\mathscr{B} \otimes \ldots \otimes T_X\mathscr{B}}_{p \text{ copies}} \otimes \underbrace{T^*_X\mathscr{B} \otimes \ldots \otimes T^*_X\mathscr{B}}_{q \text{ copies}}\,. \tag{30}$$

According to (4), $\mathscr{T}^0_1 = T^*_X\mathscr{B}$ and $\mathscr{T}^1_0 = T_X\mathscr{B}$. With n being the dimension of the underlying vector space $T_X\mathscr{B}$, the dimension of $\mathscr{T}^p_q$ is $\dim \mathscr{T}^p_q = n^O$, where O is the order of the tensor space according to (26).

It can be shown [9] that any tensor $\mathbf{T} \in \mathscr{T}^p_q$ has the unique representation

$$\mathbf{T} = \mathrm{T}^{A_1 \ldots A_p}{}_{B_1 \ldots B_q} \vec{\mathbf{E}}_{A_1} \otimes \ldots \otimes \vec{\mathbf{E}}_{A_p} \otimes \vec{\boldsymbol{\varepsilon}}^{B_1} \otimes \ldots \otimes \vec{\boldsymbol{\varepsilon}}^{B_q} \tag{31}$$

in terms of its components defined in (24). Here summation is implied for all repeated indices over the range from 1 to n.

2.3 Dual spaces, scalar products, and dual tensors

The tensor space $\mathscr{T}^p_q$ being a vector space itself, the dual space $\mathscr{T}^{*p}_q$ is the space $\mathscr{L}(\mathscr{T}^p_q; \mathbb{R})$. It follows from (30) that

$$\mathscr{T}^{*p}_q = \underbrace{T^*_X\mathscr{B} \otimes \ldots \otimes T^*_X\mathscr{B}}_{p \text{ copies}} \otimes \underbrace{T_X\mathscr{B} \otimes \ldots \otimes T_X\mathscr{B}}_{q \text{ copies}}\,. \tag{32}$$

Furthermore, the dual of $\mathscr{T}^{*p}_q$ is $\mathscr{T}^p_q$. Duality of two tensors is illustrated as follows. Consider two tensors $\mathbf{R}$ and $\mathbf{S}$ which live in dual tensor spaces, i.e. $\mathbf{R} \in \mathscr{T}^p_q$ and $\mathbf{S} \in \mathscr{T}^{*p}_q$. With their following component representations,

$$\mathbf{R} = R^{A_1 \ldots A_p}{}_{B_1 \ldots B_q} \vec{\mathbf{E}}_{A_1} \otimes \ldots \otimes \vec{\mathbf{E}}_{A_p} \otimes \vec{\boldsymbol{\varepsilon}}^{B_1} \otimes \ldots \otimes \vec{\boldsymbol{\varepsilon}}^{B_q} \in \mathscr{T}^p_q\,, \tag{33}$$

$$\mathbf{S} = S_{C_1 \ldots C_p}{}^{D_1 \ldots D_q} \vec{\boldsymbol{\varepsilon}}^{C_1} \otimes \ldots \otimes \vec{\boldsymbol{\varepsilon}}^{C_p} \otimes \vec{\mathbf{E}}_{D_1} \otimes \ldots \otimes \vec{\mathbf{E}}_{D_q} \in \mathscr{T}^{*p}_q\,, \tag{34}$$

their scalar product yields

$$\langle \mathbf{R}, \mathbf{S} \rangle_X = R^{A_1 \ldots A_p}{}_{B_1 \ldots B_q} S_{A_1 \ldots A_p}{}^{B_1 \ldots B_q}\,. \tag{35}$$

Remark 2.10. The property (32) is a special case of a general natural isomorphism (see e.g. [6]). Let $\mathscr{U}$ and $\mathscr{V}$ be two vector spaces with duals $\mathscr{U}^*$ and $\mathscr{V}^*$, respectively. The dual space $(\mathscr{U} \otimes \mathscr{V})^*$ of the space $\mathscr{U} \otimes \mathscr{V}$ is naturally isomorphic to $\mathscr{U}^* \otimes \mathscr{V}^*$, i.e.

$$(\mathscr{U} \otimes \mathscr{V})^* \simeq \mathscr{U}^* \otimes \mathscr{V}^*\,. \quad \square \tag{36}$$

Tensors can be used to represent mappings between vector spaces. We shall illustrate this for second-order tensors, as these are of special importance in continuum mechanics. We first introduce *simple* second-order tensors by tensor products of vectors and one-forms, respectively.

Definition 2.15: Let $\vec{\mathbf{U}}, \vec{\mathbf{V}} \in T_X\mathscr{B}$ and $\vec{\boldsymbol{\alpha}}, \vec{\boldsymbol{\beta}} \in T_X^*\mathscr{B}$. Then construct the following *simple tensors*:

$$\vec{\mathbf{U}} \otimes \vec{\boldsymbol{\alpha}} \in T_X\mathscr{B} \otimes T_X^*\mathscr{B} = \mathscr{T}^{1\cdot}_{\cdot 1}, \qquad \vec{\boldsymbol{\alpha}} \otimes \vec{\mathbf{U}} \in T_X^*\mathscr{B} \otimes T_X\mathscr{B} = \mathscr{T}^{\cdot 1}_{1\cdot}, \tag{37, 38}$$

$$\vec{\boldsymbol{\alpha}} \otimes \vec{\boldsymbol{\beta}} \in T_X^*\mathscr{B} \otimes T_X^*\mathscr{B} = \mathscr{T}^{0}_{2}, \qquad \vec{\mathbf{U}} \otimes \vec{\mathbf{V}} \in T_X\mathscr{B} \otimes T_X\mathscr{B} = \mathscr{T}^{2}_{0}, \tag{39, 40}$$

such that

$$\vec{\mathbf{U}} \otimes \vec{\boldsymbol{\alpha}} : T_X\mathscr{B} \to T_X\mathscr{B} : \vec{\mathbf{V}} \mapsto \vec{\mathbf{U}} \otimes \vec{\boldsymbol{\alpha}}\vec{\mathbf{V}} = \vec{\mathbf{U}}\langle\vec{\boldsymbol{\alpha}}, \vec{\mathbf{V}}\rangle_X, \tag{41}$$

$$\vec{\boldsymbol{\alpha}} \otimes \vec{\mathbf{U}} : T_X^*\mathscr{B} \to T_X^*\mathscr{B} : \vec{\boldsymbol{\beta}} \mapsto \vec{\boldsymbol{\alpha}} \otimes \vec{\mathbf{U}}\vec{\boldsymbol{\beta}} = \vec{\boldsymbol{\alpha}}\langle\vec{\mathbf{U}}, \vec{\boldsymbol{\beta}}\rangle_X, \tag{42}$$

$$\vec{\boldsymbol{\alpha}} \otimes \vec{\boldsymbol{\beta}} : T_X\mathscr{B} \to T_X^*\mathscr{B} : \vec{\mathbf{U}} \mapsto \vec{\boldsymbol{\alpha}} \otimes \vec{\boldsymbol{\beta}}\vec{\mathbf{U}} = \vec{\boldsymbol{\alpha}}\langle\vec{\boldsymbol{\beta}}, \vec{\mathbf{U}}\rangle_X, \tag{43}$$

$$\vec{\mathbf{U}} \otimes \vec{\mathbf{V}} : T_X^*\mathscr{B} \to T_X\mathscr{B} : \vec{\boldsymbol{\alpha}} \mapsto \vec{\mathbf{U}} \otimes \vec{\mathbf{V}}\vec{\boldsymbol{\alpha}} = \vec{\mathbf{U}}\langle\vec{\mathbf{V}}, \vec{\boldsymbol{\alpha}}\rangle_X. \quad \square \tag{44}$$

Remark 2.11: In Definition 2.15 we defined second-order tensors as linear mappings between vector spaces thus deviating from the more general Definition 2.10, where we defined tensors as linear maps of Cartesian products of vector spaces on the space of real numbers. A discussion of the relations between Definition 2.10 and Definition 2.15 is given in [9, p. 339]. □

Remark 2.12: In a more abstract setting we can say that simple tensors map a vector space $\mathscr{U}$ on a vector space $\mathscr{V}$. Let $u \in \mathscr{U}$, $v \in \mathscr{V}$ and $w \in \mathscr{W}$; then we have for instance

$$v \otimes w : \mathscr{U} \to \mathscr{V}, \tag{45}$$

where according to Definition 2.15 the spaces $\mathscr{U}$ and $\mathscr{W}$ have to be chosen such that a scalar product can be formed of arbitrary elements $u \in \mathscr{U}$ and $w \in \mathscr{W}$. In other words, $\mathscr{U}$ and $\mathscr{W}$ must be dual spaces, i.e. $\mathscr{W} = \mathscr{U}^*$, so that $v \otimes w \in \mathscr{V} \otimes \mathscr{U}^*$. Note that on the other hand, $v \otimes w$ also serves as a map

$$v \otimes w : \mathscr{V} \times \mathscr{U}^* \to \mathbf{R}, \tag{46}$$

in accordance with (29). In fact, it may be shown (e.g. [6]) that there is a natural isomorphism

$$\mathscr{L}(\mathscr{U}; \mathscr{V}) \simeq \mathscr{V} \otimes \mathscr{U}^* \tag{47}$$

for any two vector spaces $\mathscr{U}$ and $\mathscr{V}$ (the expressions (27) and (30) are a special case of this). In a manifold setting and for second-order tensors, $\mathscr{U}$ and $\mathscr{U}^*$ are identified either with $T_X\mathscr{B}$ and $T_X^*\mathscr{B}$, respectively, or with $T_X^*\mathscr{B}$ and $T_X\mathscr{B}$, respectively. However, we can identify the space $\mathscr{V}$ either with $T_X\mathscr{B}$ or with $T_X^*\mathscr{B}$ arbitrarily. □

Next, we introduce general second-order tensors. Recall that $\vec{\mathbf{E}}_A$ and $\vec{\boldsymbol{\varepsilon}}^A$ are the dual basises of the spaces $T_X\mathscr{B}$ and $T_X^*\mathscr{B}$, respectively. We can define the following *second-order* tensors.

Definition 2.16:

$$\mathbf{T}^{\backslash} \in T_X\mathscr{B} \otimes T_X^*\mathscr{B} = \mathscr{T}^{1\cdot}_{\cdot 1}, \qquad \mathbf{T}^{\backslash} = \mathrm{T}^A{}_B\vec{\mathbf{E}}_A \otimes \vec{\boldsymbol{\varepsilon}}^B, \tag{48}$$

$$\mathbf{T}^{/} \in T_X^*\mathscr{B} \otimes T_X\mathscr{B} = \mathscr{T}^{\cdot 1}_{1\cdot}, \qquad \mathbf{T}^{/} = \mathrm{T}_A{}^B\vec{\boldsymbol{\varepsilon}}^A \otimes \vec{\mathbf{E}}_B, \tag{49}$$

$$\mathbf{T}^{\flat} \in T_X^*\mathscr{B} \otimes T_X^*\mathscr{B} = \mathscr{T}^{0}_{2}, \qquad \mathbf{T}^{\flat} = \mathrm{T}_{AB}\vec{\boldsymbol{\varepsilon}}^A \otimes \vec{\boldsymbol{\varepsilon}}^B, \tag{50}$$

$$\mathbf{T}^{\sharp} \in T_X\mathscr{B} \otimes T_X\mathscr{B} = \mathscr{T}^{2}_{0}, \qquad \mathbf{T}^{\sharp} = \mathrm{T}^{AB}\vec{\mathbf{E}}_A \otimes \vec{\mathbf{E}}_B. \quad \square \tag{51}$$

Definitions 2.15 and 2.16 lead to the following equations:

$$\mathbf{T}^{\backslash} : T_X\mathscr{B} \to T_X\mathscr{B} : \vec{\mathbf{U}} \mapsto \mathbf{T}^{\backslash}\vec{\mathbf{U}} = \vec{\mathbf{V}}, \qquad \mathrm{T}^A{}_B U^B = V^A, \tag{52}$$

$$\mathbf{T}^{/} : T_X^*\mathscr{B} \to T_X^*\mathscr{B} : \vec{\boldsymbol{\alpha}} \mapsto \mathbf{T}^{/}\vec{\boldsymbol{\alpha}} = \vec{\boldsymbol{\beta}}, \qquad \mathrm{T}_A{}^B \alpha_B = \beta_A, \tag{53}$$

$$\mathbf{T}^{\flat} : T_X\mathscr{B} \to T_X^*\mathscr{B} : \vec{\mathbf{U}} \mapsto \mathbf{T}^{\flat}\vec{\mathbf{U}} = \vec{\boldsymbol{\alpha}}, \qquad \mathrm{T}_{AB} U^B = \alpha_A, \tag{54}$$

$$\mathbf{T}^{\sharp} : T_X^*\mathscr{B} \to T_X\mathscr{B} : \vec{\boldsymbol{\alpha}} \mapsto \mathbf{T}^{\sharp}\vec{\boldsymbol{\alpha}} = \vec{\mathbf{U}}, \qquad \mathrm{T}^{AB} \alpha_B = U^A, \tag{55}$$

in terms of the components of $\vec{\mathbf{U}}, \vec{\mathbf{V}}, \vec{\boldsymbol{\alpha}}, \vec{\boldsymbol{\beta}}$: $\vec{\mathbf{U}} = U^A\vec{\mathbf{E}}_A$, $\vec{\mathbf{V}} = V^A\vec{\mathbf{E}}_A$, $\vec{\boldsymbol{\alpha}} = \alpha_A\vec{\boldsymbol{\varepsilon}}^A$, $\vec{\boldsymbol{\beta}} = \beta_A\vec{\boldsymbol{\varepsilon}}^A$.

Remark 2.13: The symbols $^{\backslash}, ^{/}$, $\flat$, and $\sharp$ indicate the positions of the tensor spaces $T_X\mathscr{B}$ and $T_X^*\mathscr{B}$, respectively, in the tensor products in equations (48) to (51). Therefore, they also can be considered as reminders of the position of covariant and contravariant indices in the components presented in (52) to (55). □

Having introduced the dual of a vector or tensor space, we can define the dual or adjoint of tensors [6]. In doing so we regard tensors as mappings between vector spaces. Let $\mathbf{T}$ be a linear map between two vector spaces $\mathscr{U}$ and $\mathscr{V}$, i.e. $\mathbf{T} : \mathscr{U} \to \mathscr{V}$. Here $\mathscr{U}$ is the domain of the linear map $\mathbf{T}$, and $\mathscr{V}$ is the range. The *dual* map $\mathbf{T}^*$ is defined as a linear map from the dual $\mathscr{V}^*$ of the range space $\mathscr{V}$ to the dual $\mathscr{U}^*$ of the domain $\mathscr{U}$ of the original map, i.e. $\mathbf{T}^* : \mathscr{V}^* \to \mathscr{U}^*$. We shall limit ourselves here to giving the definition for second-order tensors (see Definition 2.16).

Definition 2.17: Let $\mathbf{T}^\backslash$, $\mathbf{T}^/$, $\mathbf{T}^\flat$, $\mathbf{T}^\sharp$ be second-order tensors; then for each of these tensors its *dual* can be defined as follows:

$$\mathbf{T}^{\backslash *} : T^*_X\mathscr{B} \to T^*_X\mathscr{B}\,, \quad \text{such that} \quad \langle \mathbf{T}^\backslash \vec{\mathbf{U}},\, \vec{\boldsymbol{\alpha}}\rangle_X = \langle \vec{\mathbf{U}},\, \mathbf{T}^{\backslash *}\vec{\boldsymbol{\alpha}}\rangle_X \quad \text{for all} \quad \vec{\mathbf{U}} \in T_X\mathscr{B} \text{ and all } \vec{\boldsymbol{\alpha}} \in T^*_X\mathscr{B}\,, \tag{56}$$

$$\mathbf{T}^{/ *} : T_X\mathscr{B} \to T_X\mathscr{B}\,, \quad \text{such that} \quad \langle \mathbf{T}^/ \vec{\boldsymbol{\alpha}},\, \vec{\mathbf{U}}\rangle_X = \langle \vec{\boldsymbol{\alpha}},\, \mathbf{T}^{/ *}\vec{\mathbf{U}}\rangle_X \quad \text{for all} \quad \vec{\boldsymbol{\alpha}} \in T^*_X\mathscr{B} \text{ and all } \vec{\mathbf{U}} \in T_X\mathscr{B}\,, \tag{57}$$

$$\mathbf{T}^{\flat *} : T_X\mathscr{B} \to T^*_X\mathscr{B}, \quad \text{such that} \quad \langle \mathbf{T}^\flat \vec{\mathbf{U}},\, \vec{\mathbf{V}}\rangle_X = \langle \vec{\mathbf{U}},\, \mathbf{T}^{\flat *}\vec{\mathbf{V}}\rangle_X \quad \text{for all} \quad \vec{\mathbf{U}},\, \vec{\mathbf{V}} \in T_X\mathscr{B}\,, \tag{58}$$

$$\mathbf{T}^{\sharp *} : T^*_X\mathscr{B} \to T_X\mathscr{B}\,, \quad \text{such that} \quad \langle \mathbf{T}^\sharp \vec{\boldsymbol{\alpha}},\, \vec{\boldsymbol{\beta}}\rangle_X = \langle \vec{\boldsymbol{\alpha}},\, \mathbf{T}^{\sharp *}\vec{\boldsymbol{\beta}}\rangle_X \quad \text{for all} \quad \vec{\boldsymbol{\alpha}},\, \vec{\boldsymbol{\beta}} \in T^*_X\mathscr{B}\,. \quad \square \tag{59}$$

It can be shown [6] that these dual tensors have the following component representations:

$$\mathbf{T}^{\backslash *} \in T^*_X\mathscr{B} \otimes T_X\mathscr{B}\,, \qquad \mathbf{T}^{\backslash *} = \mathrm{T}^*{}_A{}^B \vec{\boldsymbol{\varepsilon}}^A \otimes \vec{\mathbf{E}}_B\,, \qquad \mathrm{T}^*{}_A{}^B = \mathrm{T}^B{}_A\,, \tag{60}$$

$$\mathbf{T}^{/ *} \in T_X\mathscr{B} \otimes T^*_X\mathscr{B}\,, \qquad \mathbf{T}^{/ *} = \mathrm{T}^{*A}{}_B \vec{\mathbf{E}}_A \otimes \vec{\boldsymbol{\varepsilon}}^B\,, \qquad \mathrm{T}^{*A}{}_B = \mathrm{T}_B{}^A\,, \tag{61}$$

$$\mathbf{T}^{\flat *} \in T^*_X\mathscr{B} \otimes T^*_X\mathscr{B}\,, \qquad \mathbf{T}^{\flat *} = \mathrm{T}^*{}_{AB} \vec{\boldsymbol{\varepsilon}}^A \otimes \vec{\boldsymbol{\varepsilon}}^B\,, \qquad \mathrm{T}^*{}_{AB} = \mathrm{T}_{BA}\,, \tag{62}$$

$$\mathbf{T}^{\sharp *} \in T_X\mathscr{B} \otimes T_X\mathscr{B}\,, \qquad \mathbf{T}^{\sharp *} = \mathrm{T}^{*AB} \vec{\mathbf{E}}_A \otimes \vec{\mathbf{E}}_B\,, \qquad \mathrm{T}^{*AB} = \mathrm{T}^{BA}\,. \tag{63}$$

Remark 2.14: Thus, we see that the matrix of components of the dual tensor is identical to the transpose of the component matrix of the original tensor, e.g. $[\mathrm{T}^*{}_A{}^B] = [\mathrm{T}^B{}_A]^\mathsf{T}$. Note that the dual of a tensor in this way has the same representation as the transpose of a tensor in standard tensor algebra. $\square$

In general, for any two maps $\mathbf{R} : \mathscr{U} \to \mathscr{V}$ and $\mathbf{S} : \mathscr{V} \to \mathscr{W}$ the dual of the composite map

$$\mathbf{T} = \mathbf{RS} \in \mathscr{W} \otimes \mathscr{U}^* \tag{64}$$

is

$$\mathbf{T}^* = \mathbf{S}^*\mathbf{R}^* \in \mathscr{U}^* \otimes \mathscr{W}\,. \tag{65}$$

Remark 2.15: Notice that the space of dual tensors is not necessarily identical to the dual tensor space. For instance, the dual of the tensor $\mathbf{T} \in \mathscr{V} \otimes \mathscr{U}^*$ is the tensor $\mathbf{T}^*$ which belongs to the tensor space $\mathscr{U}^* \otimes \mathscr{V}$, whereas according to (36), $\mathscr{V}^* \otimes \mathscr{U}$ is the dual space to $\mathscr{V} \otimes \mathscr{U}^*$. Only when $\mathscr{U} = \mathscr{V}$ are both spaces naturally isomorphic; for second-order tensors this condition is met only for mixed tensors. $\square$

It is noted that $\mathbf{T}^\flat$ and $\mathbf{T}^{\flat *}$ belong to the same tensor space $\mathscr{T}^0_2$; likewise, $\mathbf{T}^\sharp$ and $\mathbf{T}^{\sharp *}$ both belong to $\mathscr{T}^2_0$. For such tensors, we can introduce the notion of self-duality.

Definition 2.18: A map $\mathbf{T}^\flat : T_X\mathscr{B} \to T^*_X\mathscr{B}$ is called *self-dual* or *self-adjoint* if

$$\mathbf{T}^\flat \equiv \mathbf{T}^{\flat *}\,. \tag{66}$$

Similarly, *self-duality of* $\mathbf{T}^\sharp : T^*_X\mathscr{B} \to T_X\mathscr{B}$ is defined as $\mathbf{T}^\sharp \equiv \mathbf{T}^{\sharp *}$. $\square$

Remark 2.16: Note that self-duality can be defined for any tensor of type $\mathscr{U} \otimes \mathscr{U}$, where $\mathscr{U}$ can be any tensor space of type $\mathscr{T}^p_q$. Hence, for second-order tensors self-duality can be defined for covariant or contravariant tensors only. For mixed second-order tensors the notion of self-duality does not make sense. $\square$

The above ideas are readily generalized to other tensor spaces. With a view on applications in continuum mechanics, we explicitly mention the fourth-order tensor $\mathbf{C}^{//} \in \mathscr{T}^{\cdot 1}_{1\cdot} \otimes \mathscr{T}^{\cdot 1}_{1\cdot} = T^*_X\mathscr{B} \otimes T_X\mathscr{B} \otimes T^*_X\mathscr{B} \otimes T_X\mathscr{B}$ that serves as the following mapping:

$$\mathbf{C}^{//} : \mathscr{T}^{1\cdot}_{\cdot 1} \to \mathscr{T}^{\cdot 1}_{1\cdot} : \mathbf{T}^\backslash \mapsto \mathbf{T}^/ = \mathbf{C}^{//}\mathbf{T}^\backslash\,. \tag{67}$$

Remark 2.17: Fourth-order tensors are denoted by boldface Roman uppercase kernel letters and a combination of two of the symbols $\backslash, /$, $\flat$, and $\sharp$. In the same sense higher order tensors of even order can be characterized. $\square$

For later reference in Section 3.2 we give the component representation of (67),

$$\begin{aligned}\mathbf{C}^{//}\mathbf{T}^\backslash &= (\mathrm{C}_A{}^B{}_C{}^D \vec{\boldsymbol{\varepsilon}}^A \otimes \vec{\mathbf{E}}_B \otimes \underbrace{\vec{\boldsymbol{\varepsilon}}^C}_{1} \otimes \underbrace{\vec{\mathbf{E}}_D}_{2})\, (T^R{}_S \underbrace{\vec{\mathbf{E}}_R}_{1} \otimes \underbrace{\vec{\boldsymbol{\varepsilon}}^S}_{2}) \\ &= (\mathrm{C}_A{}^B{}_C{}^D T^R{}_S \langle \vec{\boldsymbol{\varepsilon}}^C,\, \vec{\mathbf{E}}_R\rangle_X\, \langle \vec{\mathbf{E}}_D,\, \vec{\boldsymbol{\varepsilon}}^S\rangle_X\, \vec{\boldsymbol{\varepsilon}}^A \otimes \vec{\mathbf{E}}_B = \mathrm{C}_A{}^B{}_C{}^D T^C{}_D \vec{\boldsymbol{\varepsilon}}^A \otimes \vec{\mathbf{E}}_B\,.\end{aligned} \tag{68}$$

Remark 2.18: The underbraced numbers indicate how scalar products in the second line of (68) are formed.

Similar to (56), the *dual of* $\mathbf{C}^{//}$ is defined through

$$\langle \mathbf{C}^{//}\mathbf{R}^\backslash,\, \mathbf{S}^\backslash\rangle_X = \langle \mathbf{R}^\backslash,\, \mathbf{C}^{//*}\mathbf{S}^\backslash\rangle_X \quad \text{for all} \quad \mathbf{R}^\backslash,\, \mathbf{S}^\backslash \in \mathscr{T}^{1\cdot}_{\cdot 1}\,, \tag{69}$$

and this tensor is self-dual if $\mathbf{C}^{//*} \equiv \mathbf{C}^{//}$.

We finally introduce here the notion of a gradient of a scalar-valued function of a tensor. For this purpose we first define the *directional derivative* of such a function.

Definition 2.19: Let $f : \ldots \times \mathscr{T}^p_q \times \ldots \to \mathbb{R}$ be a function and $\mathbf{S}, \mathbf{T} \in \mathscr{T}^p_q$, then the *directional derivative* at $\mathbf{S}$ *in the direction* $\mathbf{T}$ is defined as

$$\mathrm{D}f(\mathbf{S})\,\mathbf{T} := \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\, f(\mathbf{S} + \varepsilon\mathbf{T})\Big|_{\varepsilon=0} . \quad \square \tag{70}$$

Definition 2.20: The *partial derivative* or *gradient* of the function $f : \ldots \times \mathscr{T}^p_q \times \ldots \to \mathbb{R}$ with respect to one of its arguments $\mathbf{T} \in \mathscr{T}^p_q$ is a linear map

$$\frac{\partial f}{\partial \mathbf{T}} : \mathscr{T}^p_q \to \mathbb{R}, \tag{71}$$

such that (see [10])

$$\langle \partial f/\partial \mathbf{T}, \mathbf{T}\rangle_X = \mathrm{D}f(\mathbf{T})\,\mathbf{T} . \quad \square \tag{72}$$

Remark 2.19: Note that $\partial f/\partial \mathbf{T}$ is a member of the space $\mathscr{T}^{*p}_q$, dual to the space $\mathscr{T}^p_q$ where the argument $\mathbf{T}$ is living in. $\square$

2.4 Inner products and transposes of tensors

In mechanics we want to measure the magnitude of vectors. For this purpose we endow the manifold $\mathscr{B}$ with a *Riemannian metric.*

Definition 2.21: A *Riemannian metric* $\mathbf{G}(X)$ on $\mathscr{B}$ is a C^∞ tensor field such that
1. $\mathbf{G}(X)$ is symmetric, i.e., for $\vec{\mathbf{V}}_1, \vec{\mathbf{V}}_2 \in T_X\mathscr{B}$ we have $\mathbf{G}(\vec{\mathbf{V}}_1, \vec{\mathbf{V}}_2) = \mathbf{G}(\vec{\mathbf{V}}_2, \vec{\mathbf{V}}_1)$;
2. $\mathbf{G}(X)$ is positive-definite, i.e., $\mathbf{G}(\vec{\mathbf{V}}, \vec{\mathbf{V}}) > 0$ for all $\vec{\mathbf{V}} \in T_X\mathscr{B}$ with $\vec{\mathbf{V}} \neq \mathbf{0}$. $\square$

Remark 2.20: If there is no possibility of confusion, we shall just write $\mathbf{G}$ without explicitly stating that it is the metric at $X \in \mathscr{B}$.

The metric $\mathbf{G}(X)$ induces an *inner product* on $\mathscr{B}$.

Definition 2.22: The *inner product on the tangent space of* $\mathscr{B}$ *at* X is the mapping

$$\vec{\mathbf{U}} \cdot \vec{\mathbf{V}} : T_X\mathscr{B} \times T_X\mathscr{B} \to \mathbb{R}, \tag{73}$$

such that

$$\vec{\mathbf{U}} \cdot \vec{\mathbf{V}} := \mathbf{G}(\vec{\mathbf{U}}, \vec{\mathbf{V}}) . \quad \square \tag{74}$$

This inner product operates on Cartesian products formed on the tangent space at X, so that we can interpret $\mathbf{G}$ also as a tensor $\mathbf{G} \in \mathscr{T}^0_2$. On the other hand, in view of (50) and (54), we can make the following interpretation:

$$\mathbf{G}^\flat : T_X\mathscr{B} \to T^*_X\mathscr{B} : \mathbf{G}^\flat \equiv \mathbf{G} . \tag{75}$$

This interpretation leads to the definition of the *inverse mapping*

$$\mathbf{G}^\sharp : T^*_X\mathscr{B} \to T_X\mathscr{B} : \mathbf{G}^\sharp := \mathbf{G}^{\flat\,-1} . \tag{76}$$

Thus, the metric $\mathbf{G}(X)$ induces the field $\mathbf{G}^\sharp(X) \in \mathscr{T}^2_0$ on $\mathscr{B}$, with which we can define another inner product, namely one on the cotangent space of $\mathscr{B}$.

Definition 2.23: The *inner product on the cotangent space* $T^*_X\mathscr{B}$ *of* $\mathscr{B}$ *at* X is the mapping

$$\vec{\boldsymbol{\alpha}} \cdot \vec{\boldsymbol{\beta}} : T^*_X\mathscr{B} \times T^*_X\mathscr{B} \to \mathbb{R}, \tag{77}$$

such that

$$\vec{\boldsymbol{\alpha}} \cdot \vec{\boldsymbol{\beta}} := \mathbf{G}^\sharp(\vec{\boldsymbol{\alpha}}, \vec{\boldsymbol{\beta}}) . \quad \square \tag{78}$$

Remark 2.21: We denote inner products by $(\cdot)$ irrespective of the space for which this inner product is defined. $\square$

Remark 2.22: Symmetry of the metric $\mathbf{G}^\flat$ according to Definition 2.21, and hence the symmetry of $\mathbf{G}^\sharp$, finds expression in the fact that both $\mathbf{G}^\flat$ and $\mathbf{G}^\sharp$ are self-dual. $\square$

Denote the components of the metric tensor $\mathbf{G}(X)$ by $\mathrm{G}_{AB}(X)$. Then the component form of (74) is

$$\vec{\mathbf{U}} \cdot \vec{\mathbf{V}} = \mathrm{G}_{AB} U^A V^B . \tag{79}$$

Correspondingly, the component form of (78) is

$$\vec{\boldsymbol{\alpha}} \cdot \vec{\boldsymbol{\beta}} = \mathrm{G}^{AB}\alpha_A\beta_B\,, \tag{80}$$

where G^{AB} is the inverse of the matrix G_{AB}.

The isomorphisms (75) and (76) provide a link between the inner products on $\mathscr{B}$, (74) and (78), and the scalar product (8) as follows. Let $\vec{\mathbf{U}}, \vec{\mathbf{V}} \in T_X\mathscr{B}$ and $\vec{\boldsymbol{\alpha}}, \vec{\boldsymbol{\beta}} \in T_X^*\mathscr{B}$. Then

$$\vec{\mathbf{U}} \cdot \vec{\mathbf{V}} = \langle \mathbf{G}^\flat\vec{\mathbf{U}}, \vec{\mathbf{V}}\rangle_X = \langle \vec{\mathbf{U}}, \mathbf{G}^\flat\vec{\mathbf{V}}\rangle_X\,, \qquad \vec{\boldsymbol{\alpha}} \cdot \vec{\boldsymbol{\beta}} = \langle \mathbf{G}^\sharp\vec{\boldsymbol{\alpha}}, \vec{\boldsymbol{\beta}}\rangle_X = \langle \vec{\boldsymbol{\alpha}}, \mathbf{G}^\sharp\vec{\boldsymbol{\beta}}\rangle_X\,. \tag{81), (82}$$

Next, we define transposes of second-order tensors. Let $\mathbf{T}$ be an arbitrary second-order tensor (mixed, covariant or contravariant). Assume that $\mathbf{T}$ maps a vector space $\mathscr{U}$ on vector space $\mathscr{V}$, i.e. $\mathbf{T}: \mathscr{U} \to \mathscr{V}$. Following [1, 7] we define the *transpose* as a map $\mathbf{T}^\mathsf{T}: \mathscr{V} \to \mathscr{U}$, where equality of certain inner products has to be stipulated. More explicitly, we define

Definition 2.24: Let $\mathbf{T}^\backslash$, $\mathbf{T}^/$, $\mathbf{T}^\flat$, $\mathbf{T}^\sharp$ be second-order tensors, let $\vec{\mathbf{U}}, \vec{\mathbf{V}} \in T_X\mathscr{B}$ and $\vec{\boldsymbol{\alpha}}, \vec{\boldsymbol{\beta}} \in T_X^*\mathscr{B}$, then the following transposed tensors can be defined:

$$\mathbf{T}^{\backslash\mathsf{T}}: T_X\mathscr{B} \to T_X\mathscr{B}\,, \quad \text{such that} \quad \mathbf{T}^\backslash\vec{\mathbf{U}} \cdot \vec{\mathbf{V}} = \mathbf{T}^{\backslash\mathsf{T}}\vec{\mathbf{V}} \cdot \vec{\mathbf{U}} \quad \text{for all} \quad \vec{\mathbf{U}}, \vec{\mathbf{V}} \in T_X\mathscr{B}\,, \tag{83}$$

$$\mathbf{T}^{/\mathsf{T}}: T_X^*\mathscr{B} \to T_X^*\mathscr{B}\,, \quad \text{such that} \quad \mathbf{T}^/\vec{\boldsymbol{\alpha}} \cdot \vec{\boldsymbol{\beta}} = \vec{\boldsymbol{\alpha}} \cdot \mathbf{T}^{/\mathsf{T}}\vec{\boldsymbol{\beta}} \quad \text{for all} \quad \vec{\boldsymbol{\alpha}}, \vec{\boldsymbol{\beta}} \in T_X^*\mathscr{B}\,, \tag{84}$$

$$\mathbf{T}^{\flat\mathsf{T}}: T_X^*\mathscr{B} \to T_X\mathscr{B}\,, \quad \text{such that} \quad \mathbf{T}^\flat\vec{\mathbf{U}} \cdot \vec{\boldsymbol{\alpha}} = \mathbf{T}^{\flat\mathsf{T}}\vec{\boldsymbol{\alpha}} \cdot \vec{\mathbf{U}} \quad \text{for all} \quad \vec{\mathbf{U}} \in T_X\mathscr{B} \quad \text{and all} \quad \vec{\boldsymbol{\alpha}} \in T_X^*\mathscr{B}\,, \tag{85}$$

$$\mathbf{T}^{\sharp\mathsf{T}}: T_X\mathscr{B} \to T_X^*\mathscr{B}\,, \quad \text{such that} \quad \mathbf{T}^\sharp\vec{\boldsymbol{\alpha}} \cdot \vec{\mathbf{U}} = \mathbf{T}^{\sharp\mathsf{T}}\vec{\mathbf{U}} \cdot \vec{\boldsymbol{\alpha}} \quad \text{for all} \quad \vec{\mathbf{U}} \in T_X\mathscr{B} \quad \text{and all} \quad \vec{\boldsymbol{\alpha}} \in T_X^*\mathscr{B}\,. \;\square \tag{86}$$

Remark 2.23: Note that, contrary to the dual of a map, the transpose of a map reverses domain and range of the map. As a consequence, only the transpose of mixed tensors leaves the tensor space unchanged, i.e. $\mathbf{T}^\backslash, \mathbf{T}^{\backslash\mathsf{T}} \in \mathscr{T}^{1}_{\cdot 1}$ and $\mathbf{T}^/, \mathbf{T}^{/\mathsf{T}} \in \mathscr{T}_{1}^{\cdot 1}$. □

The transpose and dual of a tensor can be linked to each other through the isomorphisms between $T_X\mathscr{B}$ and $T_X^*\mathscr{B}$ in (75) and (76). It is readily verified on the basis of (56)–(59) and (83)–(86) that

$$\mathbf{T}^{\backslash *} = \mathbf{G}^\flat\mathbf{T}^{\backslash\mathsf{T}}\mathbf{G}^\sharp, \qquad \mathbf{T}^{/*} = \mathbf{G}^\sharp\mathbf{T}^{/\mathsf{T}}\mathbf{G}^\flat\,, \tag{87), (88}$$

$$\mathbf{T}^{\flat *} = \mathbf{G}^\flat\mathbf{T}^{\flat\mathsf{T}}\mathbf{G}^\flat\,, \qquad \mathbf{T}^{\sharp *} = \mathbf{G}^\sharp\mathbf{T}^{\sharp\mathsf{T}}\mathbf{G}^\sharp\,. \tag{89), (90}$$

From (79), (80), and (83) it follows for the component representation of $\mathbf{T}^{\backslash\mathsf{T}} = \check{\mathrm{T}}^A{}_B\vec{\mathbf{E}}_A \otimes \vec{\boldsymbol{\varepsilon}}^B$, that

$$\mathrm{G}_{AB}\mathrm{T}^A{}_C U^C V^B = \check{\mathrm{T}}^R{}_B U^C V^B \mathrm{G}_{RC}\,, \tag{91}$$

and since the vectors $\vec{\mathbf{U}}$ and $\vec{\mathbf{V}}$ are arbitrary, that

$$\mathrm{G}_{AB}\mathrm{T}^A{}_C = \check{\mathrm{T}}^R{}_B\mathrm{G}_{RC}\,. \tag{92}$$

Multiplication of (92) with G^{DC} leads to

$$\check{\mathrm{T}}^A{}_B = \mathrm{G}_{BD}\mathrm{T}^D{}_C G^{CA} = \mathrm{T}_B{}^A\,. \tag{93}$$

Remark 2.24: A corresponding result has been derived for the transpose of a mixed two-point tensor in [1] (see Section 3.1), (102). □

Remark 2.25: To avoid confusion of the letter T (symbol for transposition) with a contravariant index T, we indicate transposed components by $\check{\mathrm{T}}$ instead of T^T. □

In a completely analogous manner we can derive the components of the remaining transposed tensors, and we summarize the result as follows:

$$\mathbf{T}^{\backslash\mathsf{T}} \in T_X\mathscr{B} \otimes T_X^*\mathscr{B}\,, \qquad \mathbf{T}^{\backslash\mathsf{T}} = \check{\mathrm{T}}^A{}_B\vec{\mathbf{E}}_A \otimes \vec{\boldsymbol{\varepsilon}}^B\,, \qquad \check{\mathrm{T}}^A{}_B = \mathrm{G}_{BD}\mathrm{T}^D{}_C\mathrm{G}^{CA} = \mathrm{T}_B{}^A\,, \tag{94}$$

$$\mathbf{T}^{/\mathsf{T}} \in T_X^*\mathscr{B} \otimes T_X\mathscr{B}\,, \qquad \mathbf{T}^{/\mathsf{T}} = \check{\mathrm{T}}_A{}^B\vec{\boldsymbol{\varepsilon}}^A \otimes \vec{\mathbf{E}}_B\,, \qquad \check{\mathrm{T}}_A{}^B = \mathrm{G}^{BD}\mathrm{T}_D{}^C\mathrm{G}_{CA} = \mathrm{T}^B{}_A\,, \tag{95}$$

$$\mathbf{T}^{\flat\mathsf{T}} \in T_X\mathscr{B} \otimes T_X\mathscr{B}\,, \qquad \mathbf{T}^{\flat\mathsf{T}} = \check{\mathrm{T}}^{AB}\vec{\mathbf{E}}_A \otimes \vec{\mathbf{E}}_B\,, \qquad \check{\mathrm{T}}^{AB} = \mathrm{G}^{BD}\mathrm{T}_{DC}\mathrm{G}^{CA} = \mathrm{T}^{BA}\,, \tag{96}$$

$$\mathbf{T}^{\sharp\mathsf{T}} \in T_X^*\mathscr{B} \otimes T_X^*\mathscr{B}\,, \qquad \mathbf{T}^{\sharp\mathsf{T}} = \check{\mathrm{T}}_{AB}\vec{\boldsymbol{\varepsilon}}^A \otimes \vec{\boldsymbol{\varepsilon}}^B\,, \qquad \check{\mathrm{T}}_{AB} = \mathrm{G}_{BD}\mathrm{T}^{DC}\mathrm{G}_{CA} = \mathrm{T}_{BA}\,. \tag{97}$$

Remark 2.26: Note that the components of a transposed tensor $\mathbf{T}^\mathsf{T}$ cannot be represented directly by the components of the original tensor $\mathbf{T}$ but rather by components of one of its associate tensors. This is in contrast with the components of a dual tensor $\mathbf{T}^*$, see Remark 2.14. □

Remark 2.27: We note that Kollmann and Hackenberg [11] give an alternative definition of transposition, be it for two-point tensors. This definition corresponds with our definition of dual or adjoint tensors as given above. It has some advantages in continuum mechanics which are presented in [11]. However, as we will demonstrate in Part 2 our definition for transposition of second-order tensors is stringent, since inner products and scalar products are distinguished, what is not the case in [11]. □

It has been noted that original and transpose of mixed tensors live in the same tensor space. For those tensors, we can introduce the notion of symmetry.

Definition 2.25: A map $\mathbf{T}^{\backslash} : T_X\mathscr{B} \to T_X\mathscr{B}$ is *symmetric* if

$$\mathbf{T}^{\backslash\mathsf{T}} \equiv \mathbf{T}^{\backslash}, \tag{98}$$

and is *anti-* or *skewsymmetric* if

$$\mathbf{T}^{\backslash\mathsf{T}} \equiv -\mathbf{T}^{\backslash}. \tag{99}$$

Similar definitions hold for mixed tensors of type $\mathbf{T}^{/} : T^*_X\mathscr{B} \to T^*_X\mathscr{B}$. □

Remark 2.28: We mention that symmetry of covariant and contravariant tensors does not make sense (compare Remark 2.23). □

3. Mappings between manifolds and related tensor operations

3.1 Maps between manifolds and two-point-tensors

In this section we consider maps $\varphi : \mathscr{B} \to \mathscr{S}$ which map an n-dimensional manifold $\mathscr{B}$ on an n-dimensional manifold $\mathscr{S}$. We assume that we have coordinate systems $\{X^A\}$ on $\mathscr{B}$ and coordinate systems $\{x^a\}$ on $\mathscr{S}$. On $\mathscr{B}$ we have a basis $\vec{\mathbf{E}}_A$ and its dual basis $\vec{\boldsymbol{\varepsilon}}_0{}^B$ and on $\mathscr{S}$ a basis $\vec{\mathbf{e}}_a$ and a dual basis $\vec{\boldsymbol{\varepsilon}}^b$. Furthermore, we presuppose that both manifolds are Riemannian, i.e., on $\mathscr{B}$ exists a metric tensor $\mathbf{G}$ and on $\mathscr{S}$ a metric tensor $\mathbf{g}$.

We introduce the notion of *diffeomorphism. In accordance with [9, p. 116] we give the following definition.*

Definition 3.1: A C^r *diffeomorphism* $\varphi : \mathscr{B} \to \mathscr{S}$ is a map of an n-dimensional manifold $\mathscr{B}$ on an n-dimensional manifold $\mathscr{S}$ which possesses an inverse φ^{-1}, i.e. $\varphi^{-1} : \mathscr{S} \to \mathscr{B}$, and where the map and its inverse are in class C^r. □

Remark 3.1: We have to distinguish between objects living on $\mathscr{B}$ and $\mathscr{S}$, respectively. Therefore, we denote vectors and tensors defined on $\mathscr{B}$ with uppercase kernel letters as e.g. $\vec{\mathbf{V}}$ and $\mathbf{T}$. Correspondingly, vectors and tensors defined on $\mathscr{S}$ are denoted by lowercase kernel letters as e.g. $\vec{\mathbf{v}}$ and $\mathbf{t}$. One-forms defined on $\mathscr{B}$ are denoted by a subscript 0, as e.g. $\vec{\boldsymbol{\alpha}}_0$, while one-forms living on $\mathscr{S}$ are denoted by Greek kernel letters without this subscript. □

Next, we introduce two-point tensors. In the present work we only need mixed two-point tensors of a special type. Therefore, we do not consider general two-point tensors. A detailed description of the algebra of two-point tensors is given by KOLLMANN and HACKENBERG [11]. DENG and VU-QUOC [12] have pointed out that the definition for transposition used by KOLLMANN and HACKENBERG in fact is the notion of adjoint tensors as found in the mathematical literature [6]. They give some other useful applications of two-point tensors.

Definition 3.2: A mixed two-point tensor $\mathscr{T}^{\backslash}$ over the map φ is a linear transformation

$$\mathscr{T}^{\backslash} : T_X\mathscr{B} \to T_x\mathscr{S} : \vec{\mathbf{U}} \mapsto \vec{\mathbf{v}} = \mathscr{T}^{\backslash}\vec{\mathbf{U}}. \quad \square \tag{100}$$

Remark 3.2: Two-point tensors are indicated by boldface uppercase calligraphic kernel letters. □

Definition 3.2 leads to the component representation $\mathscr{T}^{\backslash} = \mathscr{T}^a{}_A \vec{\mathbf{e}}_a \otimes \vec{\boldsymbol{\varepsilon}}_0{}^A$ with $\vec{\mathbf{e}}_a \in T_x\mathscr{S}$ and $\vec{\boldsymbol{\varepsilon}}^A \in T^*_X\mathscr{B}$.

Following [1] we generalize Definition 2.24 to the transposition of two-point tensors.

Definition 3.3: Let $\mathscr{T}^{\backslash} : T_X\mathscr{B} \to T_x\mathscr{S}$ be a two-point tensor as in (100). Then the transpose $\mathscr{T}^{\backslash\mathsf{T}}$ is defined as

$$\mathscr{T}^{\backslash\mathsf{T}} : T_x\mathscr{S} \to T_X\mathscr{B}, \quad \text{such that} \quad \mathscr{T}^{\backslash}\vec{\mathbf{U}} \cdot \vec{\mathbf{v}} = \vec{\mathbf{U}} \cdot \mathscr{T}^{\backslash\mathsf{T}}\vec{\mathbf{v}} \quad \text{for all} \quad \vec{\mathbf{U}} \in T_X\mathscr{B} \quad \text{and all} \quad \vec{\mathbf{v}} \in T_x\mathscr{S}. \tag{101}$$

By a procedure completely analogous to that presented in connection with the derivation of (93) the following component representation for the transpose of $\mathscr{T}^{\backslash\mathsf{T}}$ is obtained:

$$\mathscr{T}^{\backslash\mathsf{T}} = g_{ab}\mathscr{T}^b{}_B G^{AB} \vec{\mathbf{E}}_A \otimes \vec{\boldsymbol{\varepsilon}}^a. \quad \square \tag{102}$$

Remark 3.3: One can write $\mathscr{T}_a{}^A$ for the components of $\mathscr{T}^{\backslash\mathsf{T}}$. However, it has to be observed, that this quantity can only be computed as indicated on the right-hand side of (102). □

In correspondence to Definition 2.17 we can define *dual* or adjoint two-point tensors. Again we consider only the dual of a two-point tensor of type $\mathscr{T}^{\backslash}$.

Definition 3.4: Let $\mathscr{T}^{\backslash} : T_X\mathscr{B} \to T_x\mathscr{S}$ be a two-point tensor. Then its *dual* is defined as

$$\mathscr{T}^{\backslash *} : T^*_x\mathscr{S} \to T^*_X\mathscr{B} \quad \text{such that} \quad \langle \mathscr{T}^{\backslash}\vec{\mathbf{V}}, \vec{\boldsymbol{\alpha}} \rangle_x = \langle \vec{\mathbf{V}}, \mathscr{T}^{\backslash *}\vec{\boldsymbol{\alpha}} \rangle_X. \quad \square \tag{103}$$

Analogously to (60) we give the component representation of $\mathscr{T}^{\backslash *}$.

$$\mathscr{T}^{\backslash *} \in T_X^*\mathscr{B} \otimes T_x^*\mathscr{S}\,, \qquad \mathscr{T}^{\backslash *} = \mathscr{T}^*{}_A{}^a \vec{\boldsymbol{\varepsilon}}_0{}^A \otimes \vec{\mathbf{e}}_a\,, \tag{104}$$

where $\mathscr{T}^*{}_A{}^a = \mathscr{T}^a{}_A$.

Remark 3.4: We note that our Definition 3.3 corresponds with that given in [1, p. 48] while the definition given for transposition by [11] is that for the dual presented in Definition 3.4. □

3.2 Invariant transformations between manifolds

In this subsection we seek for transformations which leave certain scalar objects invariant under the map $\varphi : \mathscr{B} \to \mathscr{S}$. We start with the scalar product of a vector $\vec{\mathbf{V}}$ and a one-form $\vec{\boldsymbol{\alpha}}_0$. For simplicity we assume that the map φ is an at least C^1-diffeomorphism.

Definition 3.5: The *tangent of the map* $\varphi : \mathscr{B} \to \mathscr{S}$ is the map

$$T_X\varphi : T_X\mathscr{B} \to T_x\mathscr{S}\,. \quad \square \tag{105}$$

Remark 3.5: In the sequel we often denote the tangent $T_X\varphi$ of the map as

$$\mathscr{F} := T_X\varphi\,, \tag{106}$$

and it is easy to show that $\mathscr{F}$ is a two-point tensor with component representation

$$\mathscr{F} = \mathscr{F}^a{}_A \vec{\mathbf{e}}_a \otimes \vec{\boldsymbol{\varepsilon}}_0{}^A\,, \tag{107}$$

where

$$\mathscr{F}^a{}_A := \frac{\partial \varphi^a}{\partial X^A}\,. \tag{108}$$

Correspondingly, the tangent $T_x\varphi^{-1}$ of the inverse map $\varphi^{-1} : T_x\mathscr{S} \to T_X\mathscr{B}$ is denoted by

$$\mathscr{F}^{-1} := T_x\varphi^{-1} \tag{109}$$

with component representation

$$\mathscr{F}^{-1} = \mathscr{F}^{-1A}{}_a \vec{\mathbf{E}}_A \otimes \vec{\boldsymbol{\varepsilon}}^a\,. \quad \square \tag{110}$$

Remark 3.6: Strictly speaking, the tangent $\mathscr{F} \in T_x\mathscr{S} \otimes T_X\mathscr{B}$ is a mixed two-point tensor of type $\mathscr{T}^{\backslash}$. However, since $\mathscr{F}$ is the only two-point tensor introduced in this paper, we omit the apostrophe ' as no ambiguity is possible. □

Under the mapping φ a vector field $\vec{\mathbf{V}} : X \to T_X\mathscr{B}$ transforms as

$$\vec{\mathbf{v}}(x) = \mathscr{F}\vec{\mathbf{V}} \circ \varphi = \varphi_*(\vec{\mathbf{V}})\,. \tag{111}$$

Following Marsden and Hughes [1] we call the transformation (111) the *push forward* of the vector field $\vec{\mathbf{V}}(X)$ by the mapping φ. Clearly, $\vec{\mathbf{v}}(x)$ is a vector field on $\mathscr{S}$.

Definition 3.6: The inverse of the transformation (111)

$$\vec{\mathbf{V}}(X) = \mathscr{F}^{-1}\vec{\mathbf{v}} = \varphi^*(\vec{\mathbf{v}}) \tag{112}$$

is called the *pull-back* of the vector field $\vec{\mathbf{v}}(x)$ by the inverse mapping φ^{-1}. □

Remark 3.7: Generally we denote push-forwards of objects (.) living on $\mathscr{B}$ with $\varphi_*(.)$ and pull-backs of objects living on $\mathscr{S}$ with $\varphi^*(.)$. □

Next, we want to define the push-forward of a one-form field $\vec{\boldsymbol{\alpha}}_0$ under the mapping φ. Generally speaking we define push-forwards by the requirement that certain scalars remain invariant under the transformation $\varphi : \mathscr{B} \to \mathscr{S}$. In the case of the push-forward of a one-form field this scalar quantity is the scalar product of a vector field and a one-form field. Therefore, we find the push-forward of a one-form field by the postulate that

$$\langle \vec{\mathbf{V}}, \vec{\boldsymbol{\alpha}}_0 \rangle_X = \langle \varphi_*(\vec{\mathbf{V}}), \varphi_*(\vec{\boldsymbol{\alpha}}_0) \rangle_x\,. \tag{113}$$

We insert (111) into the right-hand side of (113) and apply (103) to obtain

$$\langle \varphi_*(\vec{\mathbf{V}}), \varphi_*(\vec{\boldsymbol{\alpha}}_0) \rangle_x = \langle \vec{\mathbf{V}}, \mathscr{F}^* \varphi_*(\vec{\boldsymbol{\alpha}}_0) \rangle_X\,. \tag{114}$$

From invariance of (113) follows

$$\varphi_*(\vec{\boldsymbol{\alpha}}_0) = \mathscr{F}^{*^{-1}} \vec{\boldsymbol{\alpha}}_0\,. \tag{115}$$

Conversely, the pull-back of a one-form defined on $\mathscr{S}$ is given by

$$\varphi^*(\vec{\alpha}) = \mathscr{F}^*\vec{\alpha}\,. \tag{116}$$

Remark 3.8: Whenever there is no ambiguity possible we use the terms "vector" and "one-form", respectively, for a vector field and a one-field, respectively. □

Remark 3.9: Equations (111), (112), (115), and (116) have been derived repeatedly in the literature (e.g. [1, 13]) in the following form:

$$\varphi_*(\vec{\mathbf{V}}) = \mathscr{F}\vec{\mathbf{V}}\,, \qquad \varphi^*(\vec{\mathbf{v}}) = \mathscr{F}^{-1}\vec{\mathbf{v}}\,,$$

$$\varphi_*(\vec{\alpha}_0) = \mathscr{F}^{-\mathsf{T}}\vec{\alpha}_0\,, \qquad \varphi^*(\vec{\alpha}) = \mathscr{F}^{\mathsf{T}}\vec{\alpha}\,.$$

Clearly the expressions for push-forwards and pull-backs of vectors are identical with our formulae (111) and (112). The difference between our equations for push-forwards and pull-backs of one-forms, respectively, and that ones cited stems from the fact, that in earlier work inner products and scalar products are indistinguishable. Therefore, dual or adjoint tensors coincide with transposed tensors. Indeed, when this distinction is dropped, our formulae for one-forms also reduced to the earlier published ones. □

Next, we will derive formulae for push-forwards and pull-backs, respectively, of second-order tensors. We first mention that earlier published formulae (e.g. [1, 13]) only preserve symmetry of covariant and contravariant tensors, respectively. For mixed tensors in the conventional setting symmetry does not exist.

Remark 3.10: Note that in our setting covariant and contravariant tensors can be self-dual, while mixed tensors can be symmetric. □

For applications in continuum mechanics we seek for push-forwards and pull-backs which preserve self-duality and symmetry of second-order tensors. A tool to find such preserving formulae for push-forwards and pull-backs, respectively, of second-order tensors are *quadratic forms.*

In the following we will derive the formulae for push-forwards of a mixed tensor and a covariant tensor, respectively, in an exemplaric manner. For mixed tensors, we use the *inner product* as appropriate quadratic form. Let $\vec{\mathbf{V}} \in T_X\mathscr{B}$ and $\mathbf{T}^\backslash \in T_X\mathscr{B} \otimes T_X^*\mathscr{B}$. We stipulate invariance of $\vec{\mathbf{V}} \cdot \mathbf{T}^\backslash\vec{\mathbf{V}}$ as

$$\vec{\mathbf{V}} \cdot \mathbf{T}^\backslash\vec{\mathbf{V}} = \varphi_*(\vec{\mathbf{V}} \cdot \mathbf{T}^\backslash\vec{\mathbf{V}}) = \varphi_*(\vec{\mathbf{V}}) \cdot \varphi_*(\mathbf{T}^\backslash)\, \varphi_*(\vec{\mathbf{V}})\,. \tag{117}$$

With (111) and (83), it follows for the right-hand side of (117) that

$$\varphi_*(\vec{\mathbf{V}}) \cdot \varphi_*(\mathbf{T}^\backslash)\, \varphi_*(\vec{\mathbf{V}}) = \vec{\mathbf{V}} \cdot \mathscr{F}^{\mathsf{T}}\varphi_*(\mathbf{T}^\backslash)\, \mathscr{F}\vec{\mathbf{V}}. \tag{118}$$

Then, invariance of equation (117), requires that

$$\varphi_*(\mathbf{T}^\backslash) = \mathscr{F}^{-\mathsf{T}}\mathbf{T}^\backslash\mathscr{F}^{-1}\,. \tag{119}$$

Next, we investigate the push-forward of a covariant tensor. Let $\vec{\mathbf{V}} \in T_X\mathscr{B}$ and $\mathbf{T}^\flat \in T_X^*\mathscr{B} \otimes T_X^*\mathscr{B}$. The appropriate quadratic form now is a *scalar product.* We consider invariance of $\langle\vec{\mathbf{V}}, \mathbf{T}^\flat\vec{\mathbf{V}}\rangle_X$:

$$\langle\vec{\mathbf{V}}, \mathbf{T}^\flat\vec{\mathbf{V}}\rangle_X = \langle\varphi_*(\vec{\mathbf{V}}),\ \varphi_*(\mathbf{T}^\flat)\, \varphi_*(\vec{\mathbf{V}})\rangle_x\,. \tag{120}$$

With (111) and (103) we have

$$\langle\varphi_*(\vec{\mathbf{V}}),\ \varphi_*(\mathbf{T}^\flat)\, \varphi_*(\vec{\mathbf{V}})\rangle_X = \langle\vec{\mathbf{V}},\, \mathscr{F}^*\varphi_*(\mathbf{T}^\flat)\, \mathscr{F}\vec{\mathbf{V}}\rangle_X\,, \tag{121}$$

and invariance requires

$$\varphi_*(\mathbf{T}^\flat) = \mathscr{F}^{*-1}\mathbf{T}^*\mathscr{F}^{-1}\,. \tag{122}$$

In an analogous manner, formulae for push-forwards of tensors $\mathbf{T}^\backslash$ and $\mathbf{T}^\sharp$ can be derived. Clearly, pull-backs are just the inverses of the respective push-forwards. In Table 1 we summarize the expressions for push-forwards and pull-backs of scalars, vectors, one-forms, and second-order tensors.

Remark 3.11: We comment on the push-forwards and pull-backs of second-order tensors presented in Table 1 as follows. Push-forwards and pull-backs of covariant and contravariant tensors are as in prior work (e.g. [1, 9, 13]) if one collapses the dual of the tangent operator $\mathscr{F}$ with its transpose ($\mathscr{F}^* = \mathscr{F}^{\mathsf{T}}$). However, the push-forwards and pull-backs of mixed tensors presented in Table 1 are new. Obviously, in all prior attempts to establish rules for push-forwards and pull-backs, respectively, the concept of invariance of quadratic forms based on scalar products has been exploited implicitly. Our new definitions for mixed second-order tensors have far reaching consequences in the description of kinematics of deformable continua as will be seen in part 2. □

Table 1. Push-forwards and pull-backs

	Push-forward $\varphi_*(.)$	Pull-back $\varphi^*(.)$
Scalar	$\varphi_*(S) = S \circ \varphi^{-1}$	$\varphi^*(s) = s \circ \varphi$
Vector	$\varphi_*(\vec{\mathbf{V}}) = \mathscr{F}\vec{\mathbf{V}}$	$\varphi^*(\vec{\mathbf{v}}) = \mathscr{F}^{-1}\vec{\mathbf{v}}$
One-form	$\varphi_*(\vec{\boldsymbol{\alpha}}_0) = \mathscr{F}^{*-1}\vec{\boldsymbol{\alpha}}_0$	$\varphi^*(\vec{\boldsymbol{\alpha}}) = \mathscr{F}^{*}\vec{\boldsymbol{\alpha}}$
Second-order tensors	$\varphi_*(\mathbf{T}^\sharp) = \mathscr{F}\mathbf{T}^\sharp\mathscr{F}^*$	$\varphi^*(\mathbf{t}^\sharp) = \mathscr{F}^{-1}\mathbf{t}^\sharp\mathscr{F}^{*-1}$
	$\varphi_*(\mathbf{T}^\flat) = \mathscr{F}^{*-1}\mathbf{T}^\flat\mathscr{F}^{-1}$	$\varphi^*(\mathbf{t}^\flat) = \mathscr{F}^{*}\mathbf{t}^\flat\mathscr{F}$
	$\varphi_*(\mathbf{T}^\backslash) = \mathscr{F}^{-\mathsf{T}}\mathbf{T}^\backslash\mathscr{F}^{-1}$	$\varphi^*(\mathbf{t}^\backslash) = \mathscr{F}^{\mathsf{T}}\mathbf{t}^\backslash\mathscr{F}$
	$\varphi_*(\mathbf{T}^/) = \mathscr{F}^{*^\mathsf{T}}\mathbf{T}^/\mathscr{F}^*$	$\varphi^*(\mathbf{t}^/) = \mathscr{F}^{*^{-\mathsf{T}}}\mathbf{t}^/\mathscr{F}^{*-1}$

Remark 3.12: Note that self-duality of covariant $\mathbf{T}^\flat$ and contravariant tensors $\mathbf{T}^\sharp$ are preserved under the push-forward operation in the sense, that if the tensors $\mathbf{T}^\flat$ and $\mathbf{T}^\sharp$ are self-dual, their push-forwards are also self-dual. Correspondingly, symmetry of mixed tensors $\mathbf{T}^\backslash$ and $\mathbf{T}^/$ is also preserved. The pull-backs defined in Table 1 again preserve self-duality and symmetry, respectively. □

Finally, we consider the push-forward of the fourth-order tensor $\mathbf{C}^{//}$ introduced in (67). Its push-forward is defined by invariance of the following scalar product:

$$\langle \mathbf{T}^\backslash, \mathbf{C}^{//}\mathbf{T}^\backslash\rangle_X = \langle \varphi_*(\mathbf{T}^\backslash), \varphi_*(\mathbf{C}^{//})\, \varphi_*(\mathbf{T}^\backslash)\rangle_x . \tag{123}$$

Since $\mathbf{C}^{//} \in \mathscr{T}^{\cdot 1}_{1\cdot} \otimes \mathscr{T}^{\cdot 1}_{1\cdot} = T^*_X\mathscr{B} \otimes T_X\mathscr{B} \otimes T^*_X\mathscr{B} \otimes T_X\mathscr{B}$, we can write

$$\varphi_*(\mathbf{C}^{//}) : T^*_X\mathscr{B} \otimes T_X\mathscr{B} \otimes T^*_X\mathscr{B} \otimes T_X\mathscr{B} \to T^*_x\mathscr{S} \otimes T_x\mathscr{S} \otimes T^*_x\mathscr{S} \otimes T_x\mathscr{S} . \tag{124}$$

This and Table 1 motivate the following symbolic representation of the push-forward of the fourth-order tensor:

$$\underbrace{T^*_x\mathscr{S} \otimes T_X\mathscr{B}}_{\mathscr{F}^{*\mathsf{T}}} \otimes \underbrace{T_x\mathscr{S} \otimes T^*_X\mathscr{B}}_{\mathscr{F}}\ \underbrace{T^*_X\mathscr{B} \otimes T_X\mathscr{B} \otimes T^*_X\mathscr{B} \otimes T_X\mathscr{B}}_{\mathbf{C}^{//}}\ \underbrace{T_X\mathscr{B} \otimes T^*_x\mathscr{S}}_{\mathscr{F}^{\mathsf{T}}} \otimes \underbrace{T^*_X\mathscr{B} \otimes T_x\mathscr{S}}_{\mathscr{F}^*} . \tag{125}$$

Therefore, the push-forward is given as

$$\varphi_*(\mathbf{C}^{//}) = (\mathscr{F}^{*^\mathsf{T}} \otimes \mathscr{F})\, \mathbf{C}^{//}(\mathscr{F}^\mathsf{T} \otimes \mathscr{F}^*) . \tag{126}$$

From Table 1 and (126) follows invariance of (123). Finally, we give the component representation of (126). For this purpose we need the component representation of $\mathscr{F}^{*^\mathsf{T}}$,

$$\mathscr{F}^{*^\mathsf{T}} = \mathscr{F}_a{}^A\vec{\boldsymbol{\varepsilon}}^a \otimes \vec{\mathbf{E}}_A , \tag{127}$$

where we have introduced the abbreviation

$$\mathscr{F}_a{}^A := \mathrm{g}_{ab}\mathscr{F}^b{}_B\mathrm{G}^{AB} . \tag{128}$$

From (126) follows

$$\begin{aligned}
&\varphi_*(\mathbf{C}^{//})\\
&= (\mathscr{F}_a{}^A\vec{\boldsymbol{\varepsilon}}^a \otimes \underbrace{\vec{\mathbf{E}}_A}_{1} \otimes \mathscr{F}^b{}_B\vec{\mathbf{e}}_b \otimes \underbrace{\vec{\boldsymbol{\varepsilon}}_0{}^B}_{2})\, \mathrm{C}_R{}^S{}_T{}^V\, \underbrace{\vec{\boldsymbol{\varepsilon}}_0{}^R}_{1} \otimes \underbrace{\vec{\mathbf{E}}_S}_{2} \otimes \underbrace{\vec{\boldsymbol{\varepsilon}}_0{}^T}_{3} \otimes \underbrace{\vec{\mathbf{E}}_V}_{4} (\mathscr{F}_c{}^C\vec{\boldsymbol{\varepsilon}}^C \otimes \underbrace{\vec{\mathbf{E}}_C}_{3} \otimes \mathscr{F}^d{}_D\, \underbrace{\vec{\boldsymbol{\varepsilon}}_0{}^D}_{4} \otimes \vec{\mathbf{e}}_d)\\
&= \mathscr{F}_a{}^A\mathscr{F}^b{}_B\delta^R_A\delta^B_S\mathrm{C}_R{}^S{}_T{}^V\mathscr{F}_c{}^C\mathscr{F}^d{}_D\delta^T_C\delta^D_V\vec{\boldsymbol{\varepsilon}}^a \otimes \vec{\mathbf{e}}_b \otimes \vec{\boldsymbol{\varepsilon}}^c \otimes \vec{\mathbf{e}}_d = \mathscr{F}_a{}^A\mathscr{F}^b{}_B\mathscr{F}_c{}^C\mathscr{F}^d{}_D\mathrm{C}_A{}^B{}_C{}^D\vec{\boldsymbol{\varepsilon}}^a \otimes \vec{\mathbf{e}}_b \otimes \vec{\boldsymbol{\varepsilon}}^c \otimes \vec{\mathbf{e}}_d .
\end{aligned} \tag{129}$$

Remark 3.13: The underbraced numbers indicate how scalar products as e.g. $\langle \vec{\mathbf{E}}_A, \vec{\boldsymbol{\varepsilon}}_0{}^R\rangle_X = \delta^R_A$ are formed. Compare also (68) and Remark 2.18, where similar scalar products are formed for tensors living on only one manifold. □

3.3 Objective time derivatives

First, we consider *objective* transformations of spatial tensor fields.

Definition 3.7: Let $\xi : \mathscr{S} \to \mathscr{S}'$ be a diffeomorphism and $\mathbf{t}(\vec{\mathbf{x}}) \in T_x\mathscr{S}$ a spatial tensor field of arbitrary order on $\mathscr{S}$. Further, let $\mathbf{t}'(\vec{\mathbf{x}}') \in T_x\mathscr{S}'$ be the transformed tensor field under the diffeomorphism $\xi : \vec{\mathbf{x}} \mapsto \vec{\mathbf{x}}'$. The tensor field

$\mathbf{t}(\vec{\mathbf{x}})$ transforms objectively under the transformation ξ if

$$\mathbf{t}' = \xi_*(\mathbf{t}), \tag{130}$$

where $\xi_*(\mathbf{t})$ is the push-forward of $\mathbf{t}$ under the diffeomorphism ξ. $\square$

Definition 3.8: We say that the tensor field $\mathbf{t}(\vec{\mathbf{x}})$ is *spatially covariant*, if (130) holds for every diffeomorphism $\xi : \mathscr{S} \to \mathscr{S}'$. $\square$

Special cases of diffeomorphism $\xi : \mathscr{S} \to \mathscr{S}'$ are *isometries* which leave the metric tensor $\mathbf{g}$ invariant.

Definition 3.9: An *isometry* is a map $\xi : \mathscr{S} \to \mathscr{S}$ with

$$\xi_*(\mathbf{g}) = \mathbf{g}. \quad \square \tag{131}$$

Remark 3.14: We note that rigid body motions are special cases of isometries. Let $\mathbf{q}(t) : T_x\mathscr{S} \to T_x\mathscr{S}$, $t \geq 0 \in \mathbb{R}$, be an orthogonal tensor and $\vec{\mathbf{c}}(t) \in T_x\mathscr{S}$ a vector on $\mathscr{S}$. Then the transformation

$$\vec{\mathbf{x}}' = \vec{\mathbf{c}}(t) + \mathbf{q}(t)\,\vec{\mathbf{x}} \tag{132}$$

is a rigid body motion. The intensively discussed issue of objectivity is equivalent with invariance under rigid body motions. $\square$

We first consider time derivatives on the reference configuration. Here we introduce the *material time derivative*, where we consider a fixed material particle X.

Definition 3.10: Let Ψ be an object defined on the reference configuration $\mathscr{B}$, where Ψ can be a scalar, a vector, a one-form, or a tensor of arbitrary order. The *material time derivative* of this object Ψ is defined as

$$\dot{\Psi} := \left.\frac{\partial \Psi}{\partial t}\right|_X, \tag{133}$$

where the index X indicates that the particle X is held fixed during differentiation. $\square$

Remark 3.15: Material time derivatives will be indicated by a superposed dot. $\square$

As is well known the material time derivative of any spatial tensor field $\mathbf{t}(\vec{\mathbf{x}})$ is not an objective tensor field. However, we can derive spatially covariant time derivatives of spatial tensor fields using the notion of *Lie derivatives*. Here we will only consider the Lie derivative induced by the map $\varphi : \mathscr{B} \to \mathscr{S}$.

Definition 3.11: The *Lie derivative* of an object ψ defined on the current configuration $\mathscr{S}$ is given by

$$\mathscr{L}_{\vec{\mathbf{v}}}(\psi) = \varphi_*\left(\frac{\mathrm{d}}{\mathrm{d}t}\,(\varphi^*(\psi))\right). \quad \square \tag{134}$$

Remark 3.16: In (134) $\varphi_*(.)$ and $\varphi^*(.)$ denote the push-forward operator and the pull-back operator, respectively, introduced in Section 3.2 (compare also Table 1). $\square$

We note without proof [1] that Lie derivatives of materially covariant tensors are also materially covariant, i.e., if under the diffeomorphism ξ introduced in Definition 3.7,

$$\mathbf{t}' = \xi_*(\mathbf{t}), \tag{135}$$

then

$$\mathscr{L}_{\vec{\mathbf{v}}'}(\mathbf{t}') = \xi_*(\mathscr{L}_{\vec{\mathbf{v}}}(\mathbf{t})). \tag{136}$$

Remark 3.17: We mention that there is a more general and rather abstract definition [1, p. 95] of the Lie derivative based on the concept of flow of the velocity field $\vec{\mathbf{v}}_t$. In this paper we use Definition 3.11 which is especially suitable for applications in continuum mechanics. $\square$

Applications of Lie derivatives will be given in part 2.

4. Summary

In this section we recapitulate the essential results of Part 1 and advantages of the suggested approach. As usual in tensor calculus on manifolds we distinguish between the tangent space $T_X\mathscr{B}$ and its dual $T^*_X\mathscr{B}$. From objects living in different spaces as e.g. vectors $\vec{\mathbf{V}} \in T_X\mathscr{B}$ and one-forms $\vec{\boldsymbol{\alpha}} \in T^*_X\mathscr{B}$ only scalar products can be formed. In Section 2.2 we introduce tensors on manifolds. To cover the great variety of higher order tensors we introduce the notions of the $\binom{p}{q}$-family of associate tensors (see Definition 2.8) and representative tensor of a $\binom{p}{q}$-family of associate tensors. In

Section 2.3 we introduce generalized dual spaces and scalar products formed of tensors living in such dual spaces. In Definition 2.17 we use scalar products to define dual or adjoint tensors. On manifolds with Riemannian metrics inner products can be defined between objects living in the same vector space. Preservation of inner products leads to the definition of transposed tensors (see Definition 2.24).

In Section 3.1 we consider diffeomorphisms between manifolds and introduce mixed two-point tensors (see Definition 3.2). For such tensors transposes (Definition 3.3) and duals (Definition 3.4) can be defined. Of central importance is Section 3.2, where rules for push-forwards and pull-backs of vectors, one-forms and second-order tensors are derived in a systematic manner. The push-forwards and pull-backs are obtained from invariance requirements imposed on quadratic forms. Our newly developed rules preserve duality and symmetry of second-order tensors (compare Table 1). Section 3.3 gives a short introduction to objective time derivatives which are based on Lie derivatives.

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